

# JOIN IRREDUCIBLE SEMIGROUPS

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**ABSTRACT.** We begin a systematic study of those finite semigroups that generate join irreducible members of the lattice of pseudovarieties of finite semigroups, which are important for the spectral theory of this lattice. Finite semigroups  $S$  that generate join irreducible pseudovarieties are characterized as follows: whenever  $S$  divides a direct product  $A \times B$  of finite semigroups, then  $S$  divides either  $A^n$  or  $B^n$  for some  $n \geq 1$ . We present a new operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  that preserves the property of join irreducibility, as does the dual operator, and show that iteration of these operators on any nontrivial join irreducible pseudovariety leads to an infinite hierarchy of join irreducible pseudovarieties.

We also describe all join irreducible pseudovarieties generated by a semigroup of order at most five. It turns out that there are 30 such pseudovarieties, and there is a relatively easy way to remember them. In addition, we survey most results known about join irreducible pseudovarieties to date and generalize a number of results in Chapter 7.3 of [The  $q$ -Theory of Finite Semigroups, Springer Monographs in Mathematics, Springer, Berlin, 2009].

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## 1. INTRODUCTION

In the 1970s, Eilenberg [3] highlighted the importance of  $\mathbf{PV}$ , the algebraic lattice of all pseudovarieties of finite semigroups, via his research with Schützenberger, by providing a correspondence between  $\mathbf{PV}$  and varieties of regular languages. Specifically, they proved that the lattice  $\mathbf{PV}$  is isomorphic to the algebraic lattice of varieties of regular languages; see the monograph by the second and third authors [20, Introduction] and the references therein.

The  $\mathbf{q}$ -theory of finite semigroups focuses on  $\mathbf{PV}$ , but in a different manner, and can be viewed in analogy with the classical real analysis theory of continuous and differentiable functions from  $[0, 1]$  into  $[0, 1]$ . The analogy is given by replacing  $[0, 1]$  with  $\mathbf{PV}$ , continuous functions with  $\mathbf{Cnt}(\mathbf{PV})$ , and differentiable functions with  $\mathbf{GMC}(\mathbf{PV})$ ; see [20, Chapter 2].

From a number of points of view,  $\mathbf{PV}$  is an important algebraic lattice with many interesting properties, and several theories have been developed for its investigation. For instance, the theorem of Reiterman [18] characterized pseudovarieties as exactly the classes defined by pseudoidentities. This led to the syntactic approach employed by Almeida in his work and monograph [1] that has become a fundamental tool in finite semigroup theory. Some of these results and techniques will be employed in the present article. Another important approach is the abstract spectral theory of  $\mathbf{PV}$  going back to Stone with lattice theoretic foundations going back to Birkhoff; see [20, Chapter 7].

Since  $\mathbf{PV}$  is a lattice, it is natural to investigate its elements that satisfy important lattice properties. For an element  $\ell$  in a lattice  $\mathcal{L}$ ,

- (1)  $\ell$  is *compact* if, for any  $\mathcal{X} \subseteq \mathcal{L}$ ,

$$\ell \leq \bigvee \mathcal{X} \implies \ell \leq \bigvee \mathcal{F} \text{ for some finite } \mathcal{F} \subseteq \mathcal{X};$$

- (2)  $\ell$  is *join irreducible* (ji) if, for any  $\mathcal{X} \subseteq \mathcal{L}$ ,

$$\ell \leq \bigvee \mathcal{X} \implies \ell \leq x \text{ for some } x \in \mathcal{X};$$

- (3)  $\ell$  is *finite join irreducible* (fji) if, for any finite  $\mathcal{F} \subseteq \mathcal{L}$ ,

$$\ell \leq \bigvee \mathcal{F} \implies \ell \leq x \text{ for some } x \in \mathcal{F};$$

- (4)  $\ell$  is *meet irreducible* (mi) if, for any set  $\mathcal{X} \subseteq \mathcal{L}$ ,

$$\ell \geq \bigwedge \mathcal{X} \implies \ell \geq x \text{ for some } x \in \mathcal{X};$$

- (5)  $\ell$  is *finite meet irreducible* (fmi) if, for any finite  $\mathcal{F} \subseteq \mathcal{L}$ ,

$$\ell \geq \bigwedge \mathcal{F} \implies \ell \geq x \text{ for some } x \in \mathcal{F};$$

- (6)  $\ell$  is *strictly join irreducible* (sji) if, for any set  $\mathcal{X} \subseteq \mathcal{L}$ ,

$$\ell = \bigvee \mathcal{X} \implies \ell \in \mathcal{X};$$

- (7)  $\ell$  is *strictly finite join irreducible* (sfji) if, for any finite  $\mathcal{F} \subseteq \mathcal{L}$ ,

$$\ell = \bigvee \mathcal{F} \implies \ell \in \mathcal{F};$$

- (8)  $\ell$  is *strictly meet irreducible* (smi) if, for any  $\mathcal{X} \subseteq \mathcal{L}$ ,

$$\ell = \bigwedge \mathcal{X} \implies \ell \in \mathcal{X};$$

- (9)  $\ell$  is *strictly finite meet irreducible* (sfmi) if, for any finite  $\mathcal{F} \subseteq \mathcal{L}$ ,

$$\ell = \bigwedge \mathcal{F} \implies \ell \in \mathcal{F}.$$

An *algebraic lattice* is a complete lattice that is join generated by its compact elements. The compact elements of  $\mathbf{PV}$  are the finitely generated pseudovarieties. The pseudovariety generated by a finite semigroup  $S$  is denoted by  $\langle\langle S \rangle\rangle$ . It is clear that for any  $\mathbf{V} \in \mathbf{PV}$ ,

$$\mathbf{V} = \bigvee \{ \langle\langle S \rangle\rangle \mid S \in \mathbf{V} \}.$$

Now the abstract spectral theory of a lattice is closely connected to the computation of its maximal distributive image, which is determined by the lattice's fji and fmi elements; see [20, Chapter 7] and the references therein. The fji and fmi elements of  $\mathbf{PV}$  are thus very important. The ji pseudovarieties are just the compact fji pseudovarieties, as is easy to see, so we are interested in finite semigroups that generate pseudovarieties that are fji or equivalently ji.

By abuse of terminology, we say that a finite semigroup  $S$  is *join irreducible* (ji) if the pseudovariety  $\langle\langle S \rangle\rangle$  is ji; finite semigroups that satisfy the properties in (3)–(9) are similarly defined. A finite semigroup  $S$  is ji if and only if for all finite semigroups  $T_1$  and  $T_2$ ,

$$S \prec T_1 \times T_2 \implies S \prec T_1^n \text{ or } S \prec T_2^n \text{ for some } n \geq 1,$$

where  $A \prec B$  means that  $A$  is a homomorphic image of a subsemigroup of  $B$ , and  $A^n = A \times A \times \cdots \times A$  denotes the direct product of  $n$  copies of  $A$ . For finite semigroups, there are several properties stronger than being ji: a finite semigroup  $S$  is  $\times$ -*prime* [1, Section 9.3] if for all finite semigroups  $T_1$  and  $T_2$ ,

$$S \prec T_1 \times T_2 \implies S \prec T_1 \text{ or } S \prec T_2;$$

a semigroup  $S$  is *Kovács–Newman* (KN) if whenever  $f : T \twoheadrightarrow S$  is a surjective homomorphism where  $T$  is a subsemigroup of  $T_1 \times T_2$  for some finite semigroups  $T_1$  and  $T_2$ , subdirectly embedded, then  $f$  factors through one of the projections. Semigroups that are KN have been completely classified [20, Section 7.4].

The following proper inclusions

$$\{\text{KN semigroups}\} \subsetneq \{\times\text{-prime semigroups}\} \subsetneq \{\text{ji semigroups}\}$$

are known to hold. For example, while any simple non-abelian group is KN, any cyclic group  $\mathbb{Z}_p$  of prime order  $p$  is  $\times$ -prime but not KN. The Brandt semigroup  $B_2 = \mathcal{M}^0(\{1\}, \{1, 2\}, \{1, 2\}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$  of order five is ji but not  $\times$ -prime [20, Example 7.4.3].

Since  $\mathbf{PV}$  is an algebraic lattice, it follows from a well-known theorem of Birkhoff that its smi elements constitute the unique minimal set of meet generators [20, Section 7.1]. It easily follows from Reiterman's theorem [20, Section 3.2] that each smi pseudovariety is defined by a single pseudoidentity, but not conversely. Now the reverse of the lattice  $\mathbf{PV}$  is not algebraic but is locally dually algebraic, so its sji elements constitute the unique minimal set of join generators for  $\mathbf{PV}$  [20, Section 7.2]. The sji pseudovarieties are precisely those having a unique proper maximal subpseudovariety.

Every ji pseudovariety is sji, but the converse does not hold, as demonstrated by several known examples [20, Proposition 7.3.22] and additional examples in Propositions 3.1 and 6.32. Hence ji pseudovarieties do not join generate the lattice  $\mathbf{PV}$ . This prompts the following tantalizing question.

**Question 1.1.** What do the ji elements in  $\mathbf{PV}$  join generate?

It is well known and not difficult to prove that the function

$$S \mapsto \begin{cases} 1 & \text{if } \langle\langle S \rangle\rangle \text{ is sji} \\ 0 & \text{otherwise} \end{cases}$$

on the class of finite semigroups is computable. On the other hand, it is unknown if the function

$$S \mapsto \begin{cases} 1 & \text{if } \langle\langle S \rangle\rangle \text{ is ji} \\ 0 & \text{otherwise} \end{cases}$$

on the same class is decidable.

**Question 1.2.** Is ji decidable, that is, is the above function computable?

If ji is not decidable, then a systematic study of ji seems doomed, in general. But even if ji is decidable, it is probably hopeless, in practice, to find all ji semigroups. In any case, an important step is to find methods to produce new ji semigroups and methods to identify and eliminate finite semigroups that are not ji. The present article develops several new methods. For semigroups of small order, in particular, the (Birkhoff) equational theory is crucial and is often used.

A pleasant feature of a finite semigroup  $S$  being ji is the “five for one phenomenon” related to the *exclusion class*  $\text{Excl}(S)$  of  $S$ , the class of all finite semigroups  $T$  for which  $S \notin \langle\langle T \rangle\rangle$ . Indeed, a finite semigroup  $S$  is ji if and only if  $\text{Excl}(S)$  is a pseudovariety [20, Theorem 7.1.2]. In this case,  $\text{Excl}(S)$  is mi and so is defined by a single pseudoidentity, and since  $\text{Excl}(S)$  is also sji, it has  $\text{Excl}(S) \vee \langle\langle S \rangle\rangle$  as a unique cover. Further,  $\langle\langle S \rangle\rangle \cap \text{Excl}(S)$  is the unique maximal subpseudovariety of  $\langle\langle S \rangle\rangle$ , and so  $\text{Excl}(S)$  determines  $\langle\langle S \rangle\rangle$ ; see [20, Section 7.1]. For example, the Brandt semigroup  $B_2$  is ji, the exclusion class  $\text{Excl}(B_2)$  coincides with the pseudovariety

$$\mathbf{DS} = \llbracket ((xy)^\omega (yx)^\omega (xy)^\omega)^\omega \approx (xy)^\omega \rrbracket$$

of finite semigroups whose  $\mathcal{J}$ -classes are subsemigroups [20, Example 7.3.4], and  $\langle\langle B_2 \rangle\rangle \cap \mathbf{DS} = \langle\langle B_0 \rangle\rangle$  is the unique maximal subpseudovariety of  $\langle\langle B_2 \rangle\rangle$ , where  $B_0$  is a subsemigroup of  $B_2$  of order four [4] (see Subsection 3.4 for the definition). More examples of maximal subpseudovarieties can be found in Section 5.

As mentioned earlier, a goal of the present article is to find new ji semigroups. One approach—and a very important problem in its own right—is to find new operators on  $\mathbf{PV}$  that preserves the property of being ji. The following are some known examples.

**Example 1.3.** For any semigroup  $S$ , let  $S^{\text{op}}$  denote the semigroup obtained by reversing the multiplication on  $S$ . Then the *dual operator*

$$\mathbf{V} \mapsto \mathbf{V}^{\text{op}} = \{S^{\text{op}} \mid S \in \mathbf{V}\}$$

on  $\mathbf{PV}$  preserves the property of being ji.

**Example 1.4** (See Lemma 5.2). For any semigroup  $S$ , let  $S^I$  denote the monoid obtained by adjoining an external identity element  $I$  to  $S$ , and define

$$S^\bullet = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S^I & \text{otherwise.} \end{cases}$$

Then the operator  $\langle S \rangle \mapsto \langle S^\bullet \rangle$  on  $\mathbf{PV}$  preserves the property of being ji.

Example 1.3 is not surprising; in fact, in many investigations—such as the finite basis problem for small semigroups [12]—it is common to identify  $S^{\text{op}}$  with  $S$ . The situation for the operator  $\langle S \rangle \mapsto \langle S^\bullet \rangle$ , however, can be different because it is possible that no new ji pseudovariety is produced. If  $S$  is a ji semigroup that is not a monoid, then the pseudovariety  $\langle S^\bullet \rangle = \langle S^I \rangle$  is also ji with  $\langle S \rangle \subsetneq \langle S^\bullet \rangle$ . But if  $S$  is a ji monoid, then  $\langle S^\bullet \rangle = \langle S \rangle$  is not a new example of ji pseudovariety. Note that the operator  $\langle S \rangle \mapsto \langle S^I \rangle$  does not preserve the property of being ji. For example, the cyclic group  $\mathbb{Z}_p$  of any prime order  $p$  generates a ji pseudovariety, but the pseudovariety  $\langle \mathbb{Z}_p^I \rangle$  is not ji since  $\langle \mathbb{Z}_p^I \rangle = \langle \mathbb{Z}_p \rangle \vee \langle Sl_2 \rangle$ , where  $Sl_2$  is the semilattice of order two.

On the other hand, it is possible for  $\langle S^I \rangle$  to be ji even though  $\langle S \rangle$  is not ji. For example, if  $S = Sl_2 \times R_2$ , where  $R_2$  is the right zero semigroup of order two, then the pseudovariety  $\langle S \rangle = \langle Sl_2 \rangle \vee \langle R_2 \rangle$  is not ji but the pseudovariety  $\langle S^I \rangle = \langle R_2^I \rangle$  is ji [20, Example 7.3.1].

**Remark 1.5.** It is clear that the operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{op}}$  also preserves the property of being sji, but the operator  $\langle S \rangle \mapsto \langle S^\bullet \rangle$  does not preserve this property. For instance, the pseudovariety  $\langle B_0 \rangle$  is sji while the pseudovariety  $\langle B_0^I \rangle$  is not sji; see Proposition 3.1.

Given a finite semigroup  $S$ , consider the right regular representation  $(S^\bullet, S)$  of  $S$  acting on  $S^\bullet$  by right multiplication. Then  $S^{\text{bar}}$  is defined by adding all constant maps on  $S^\bullet$  to  $S$ , where multiplication is composition with the variable written on the left. Note that if  $(S, S)$  is a faithful transformation semigroup, then we shall see later that the semigroup obtained from  $S$  by adjoining the constant mappings on  $S$  generates the same pseudovariety as  $S^{\text{bar}}$  and hence we sometimes (abusively) denote this latter semigroup by  $S^{\text{bar}}$ , as well. Some small examples of  $S^{\text{bar}}$  can be found in Section 3.

It turns out that the operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}} = \langle S^{\text{bar}} \mid S \in \mathbf{V} \rangle$  on  $\mathbf{PV}$  preserves the property of being ji. This result, the details of which are given in Subsection 4.3, is important: for any finite nontrivial ji semigroup  $S$ , the pseudovarieties

$$\langle S^{\text{bar}} \rangle, \langle (S^{\text{bar}})^{\flat} \rangle, \langle ((S^{\text{bar}})^{\flat})^{\text{bar}} \rangle, \langle (((S^{\text{bar}})^{\flat})^{\text{bar}})^{\flat} \rangle, \dots,$$

where  $X^{\flat} = ((X^{\text{op}})^{\text{bar}})^{\text{op}}$ , constitute an infinite increasing chain of ji pseudovarieties (Corollary 4.10) whose union is an fji pseudovariety that is not compact [20, Chapter 7].

Unsurprisingly, irregularities do show up when the operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  is applied. For instance, it is sometimes possible for  $\langle S^{\text{bar}} \rangle = \langle S \rangle$ , so that no new ji pseudovariety is obtained. Further, it is possible for  $\langle S^{\text{bar}} \rangle$  to be ji

even though  $\langle\langle S \rangle\rangle$  is not ji. An important class of examples will be given in Subsection 4.5.

A main result of this article is the complete classification of all semigroups of order up to five that generate ji pseudovarieties. We want to give the reader an easy way to remember the list. First, we have the three operators  $S \mapsto S^{\text{op}}$ ,  $S \mapsto S^\bullet$ , and  $S \mapsto S^{\text{bar}}$ , and their *iterations* such as  $((S^{\text{op}})^{\text{bar}})^{\text{op}}$  and  $(((((S^{\text{bar}})^{\text{op}})^\bullet)^{\text{bar}})^{\text{op}})^{\text{bar}}$ . If we have a list of ji semigroups, applying the three operators and their iterations give ji semigroups that may or may not generate new ji pseudovarieties.

A ji pseudovariety of the form  $\langle\langle T \rangle\rangle$  is said to be *primitive* if it *cannot* be obtained by applying the operators  $S \mapsto S^\bullet$  and  $S \mapsto S^{\text{bar}}$  to a semigroup that generates a ji proper subpseudovariety of  $\langle\langle T \rangle\rangle$ . Now we are only interested in knowing the primitive ji pseudovarieties up to isomorphism and anti-isomorphism of members since the others can be obtained by applying the operators. Therefore when describing ji pseudovarieties generated by semigroups of order up to five, it suffices to list, up to isomorphism and anti-isomorphism, only those that are primitive.

$n$	Semigroups of order $n$ that generate primitive ji pseudovarieties
2	$\mathbb{Z}_2, N_2, L_2$
3	$\mathbb{Z}_3, N_3$
4	$\mathbb{Z}_4, N_4, A_0$
5	$\mathbb{Z}_5, N_5, A_2, B_2, \ell_3^{\text{bar}}$

See Section 3 for the presentations and multiplication tables of these semigroups. The only new example of semigroup that generates a primitive ji pseudovariety is  $\ell_3^{\text{bar}}$ ; all the other semigroups were previously known to be ji. Note that  $\ell_3^{\text{bar}}$  is ji but  $\ell_3$  is not; see Subsection 4.5. We extend this example to an infinite family in Subsection 4.5.

The statement of the above result regarding semigroups of order up to five is straightforward, but its proof is not so; it requires advanced theory of subpseudovarieties of pseudovarieties generated by small semigroups [4, 7, 10, 11, 13, 14, 23, 24, 25], advanced algebraic theory of finite semigroups [20], and knowledge of bases of pseudoidentities for many pseudovarieties of the form  $\langle\langle S_1 \rangle\rangle \vee \langle\langle S_2 \rangle\rangle \vee \cdots \vee \langle\langle S_k \rangle\rangle$ .

The following are all other ji semigroups known to us.

**1.1. Groups.** It is an easy observation that a finite group generates a ji pseudovariety of semigroups if and only if it generates a ji pseudovariety of groups; see [20, Chapter 7]. A pseudovariety  $\mathbf{V}$  of groups is called *saturated* if whenever  $\varphi : G \rightarrow H$  is a homomorphism of finite groups with  $H \in \mathbf{V}$ , there exists a subgroup  $K \leq G$  such that  $K \in \mathbf{V}$  and  $K\varphi = H$ . It is observed in [21] that any pseudovariety of groups closed under extension is saturated. In particular, for any prime  $p$ , the pseudovariety of  $p$ -groups is saturated. It is almost immediate from the definition that if  $\mathbf{V}$  is a saturated pseudovariety of groups, then a group  $G \in \mathbf{V}$  generates a ji pseudovariety in the lattice of all semigroup pseudovarieties if and only if it generates a ji

member of the lattice of subpseudovarieties of  $\mathbf{V}$ . In particular, a  $p$ -group  $G$  is ji if and only if whenever  $G$  divides a direct product  $A \times B$  of  $p$ -groups, then  $G$  divides either  $A^n$  or  $B^n$  for some  $n \geq 1$ .

We do not know of any methods for obtaining pseudoidentities for the exclusion pseudovariety of a ji group.

*Abelian groups.* The following statements on any directly indecomposable finite abelian group  $A$  are equivalent:  $A$  is ji,  $A$  is  $\times$ -prime, and  $A \cong \mathbb{Z}_{p^n}$  for some prime  $p$  and  $n \geq 1$ . This result follows from the Fundamental Theorem of Finite Abelian Groups and that  $\mathbb{Z}_{p^n}$  *lifts* in the sense that whenever  $\mathbb{Z}_{p^n}$  is a homomorphic image of some semigroup  $S$ , then  $\mathbb{Z}_{p^{n+r}}$  embeds into  $S$  for some  $r \geq 0$ .

*Monolithic groups.* A finite group  $G$  is *monolithic* if it contains a unique minimal nontrivial normal subgroup  $N$ ; in this case,  $N$  is called the *monolith* of  $G$ , and it is well known that  $N \cong H^n$  for some simple group  $H$  and  $n \geq 1$ .

A finite group is monolithic if and only if it is subdirectly indecomposable; recall that a semigroup  $S$  is a *subdirect product* of  $S_1$  and  $S_2$ , written  $S \ll S_1 \times S_2$ , if  $S$  is a subsemigroup of  $S_1 \times S_2$  mapping onto both  $S_1$  and  $S_2$  via the projections  $\pi_i$ . A semigroup  $S$  is *subdirectly indecomposable* (sdi) if  $S \ll S_1 \times S_2$  implies that at least one of the projections  $\pi_i : S \twoheadrightarrow S_i$  is an isomorphism. Therefore when locating ji groups from among finite groups, it suffices to concentrate on those that are monolithic.

*Non-abelian monolithic groups.* Kovács and Newman proved that any monolithic group with non-abelian monolith is KN [20, Section 7.4] and so also  $\times$ -prime and ji. Therefore all simple non-abelian groups are ji.

*Abelian monolithic groups.* An abelian monolith  $N$  of a finite group  $G$  *splits* if there exists a subgroup  $K$  of  $G$  so that  $N \cap K = \{1\}$  and  $NK = G$ . A finite subdirectly indecomposable group with an abelian monolith that splits is ji; this result is due to Bergman and its proof is given in Subsection 4.6. Therefore the symmetric group  $\text{Sym}_3$  over three symbols is ji.

*Groups of small order.* The smallest groups that we do not know if they generate ji pseudovarieties are of order eight: the quaternion group  $Q_8$  and the Dihedral Group  $D_4$  of the square. Let  $G \in \{Q_8, D_4\}$ . Then forming  $G \times G$  and dividing out the two centers identified,  $(G \times G)/\{(1,1), (a,a)\}$  gives isomorphic groups, denoted by  $G \circ G$ . Since  $G \leq G \circ G$ , it follows that  $\langle\langle Q_8 \rangle\rangle = \langle\langle D_4 \rangle\rangle$ . Thus  $Q_8$  and  $D_4$  are not  $\times$ -prime and so also not KN.

**Question 1.6.** Is the pseudovariety  $\langle\langle Q_8 \rangle\rangle = \langle\langle D_4 \rangle\rangle$  ji?

For all groups of order up to seven, we know which ones are ji.

**1.2.  $\mathcal{J}$ -trivial semigroups.** Presently, the following are all known ji  $\mathcal{J}$ -trivial semigroups:

- $N_n = \langle a \mid a^n = 0 \rangle$ ,  $n \geq 1$ ;
- $H_n = \langle e, f \mid e^2 = e, f^2 = f, (ef)^n = 0 \rangle$ ,  $n \geq 1$ ;
- $K_n = \langle e, f \mid e^2 = e, f^2 = f, (ef)^n e = 0 \rangle$ ,  $n \geq 1$ ;
- $N_n^I, H_n^I, K_n^I$ ,  $n \geq 1$ .



The pseudoidentity defining the pseudovariety  $\text{Excl}(N_n)$  is given in Subsection 5.4, while the pseudovarieties  $\text{Excl}(H_n)$  and  $\text{Excl}(K_n)$  are defined by the pseudoidentities  $(x^\omega y^\omega)^{n+\omega} \approx (x^\omega y^\omega)^n$  and  $(x^\omega y^\omega)^{n+\omega} x^\omega \approx (x^\omega y^\omega)^n x^\omega$ , respectively [9, Propositions 2.3 and 3.3].

**Question 1.7.** Are there other ji pseudovarieties generated by  $\mathcal{J}$ -trivial semigroups?

We think the answer is probably yes.

**1.3. Bands.** The pseudovariety  $\mathbf{B}$  of all finite bands is fji; see Corollary 4.11. Each proper subpseudovariety of  $\mathbf{B}$  is compact and a complete description of the lattice of subpseudovarieties of  $\mathbf{B}$  is well known; see, for example, Almeida [1, Section 5.5]. Let  $\mathbf{LNB} = \mathbf{Sl} \vee \mathbf{LZ}$ ,  $\tilde{\alpha}\mathbf{V} = \mathbf{RZ} \textcircled{\mathbf{m}} \mathbf{V}$ , and  $\tilde{\beta}\mathbf{V} = \mathbf{LZ} \textcircled{\mathbf{m}} \mathbf{V}$ . Then by [16], the proper, nontrivial sji pseudovarieties of bands are as follows:

- $\mathbf{LZ}$ ,  $\mathbf{RZ}$ , and  $\mathbf{Sl}$ ;
- $(\tilde{\alpha}\tilde{\beta})^n \mathbf{Sl}$  and  $\tilde{\beta}(\tilde{\alpha}\tilde{\beta})^n \mathbf{Sl}$ ,  $n \geq 1$ , and their duals;
- $(\tilde{\beta}\tilde{\alpha})^{n+1} \mathbf{LNB}$  and  $\tilde{\alpha}(\tilde{\beta}\tilde{\alpha})^n \mathbf{LNB}$ ,  $n \geq 0$ , and their duals.

However, we observe that  $\tilde{\alpha}\mathbf{LNB} = \tilde{\alpha}\mathbf{LZ}$ . From this, and our results on  $S \mapsto S^{\text{bar}}$  preserving join irreducibility, it follows that the pseudovariety generated by a finite band is ji if and only if it is sji; see Theorem 4.13. As observed after Question 1.1, it is decidable if a finite semigroup generates a sji pseudovariety. Therefore, it is also decidable if a finite band generates a ji pseudovariety, whence Question 1.2 is positively answered for bands.

**1.4. KN semigroups.** All KN semigroups are known [20, Section 7]. These are semigroups with kernel (minimal two-sided ideal) a Rees matrix semigroup over a monolithic group with non-abelian monolith that acts faithfully on the right and left of the kernel.

**1.5. Semigroups of order six.** The new primitive ji pseudovarieties are  $\langle\langle \text{Sym}_3 \rangle\rangle$ ,  $\langle\langle N_6 \rangle\rangle$ , and  $\langle\langle K_1 \rangle\rangle$ . Investigation of semigroups of order six is still ongoing, but it seems there are no new examples.

**1.6. The subdirectly indecomposable viewpoint.** Since each finite semigroup divides (in fact, is a subdirect product of) its sdi homomorphic images, we can restrict our search for new ji semigroups to sdi semigroups, just as in the case of groups, when we can restrict our search to monolithic finite groups.

In more detail, to find the ji pseudovarieties, we clearly need only to find finite semigroups  $S$  such that  $\langle\langle S \rangle\rangle$  is ji and there exist no semigroups  $T$  with  $|T| < |S|$  and  $\langle\langle T \rangle\rangle = \langle\langle S \rangle\rangle$ . Such a semigroup  $S$  is called a *minimal order generator* for the compact pseudovariety  $\langle\langle S \rangle\rangle$ .

Now the minimal order generators of ji pseudovarieties, in fact of sji pseudovarieties, must be sdi. To see this, suppose that  $S$  is any finite semigroup that is not sdi. Then  $S \ll S_1 \times S_2 \times \cdots \times S_k$  for some homomorphic images  $S_j$  of  $S$  such that  $|S_j| < |S|$ . But since  $\langle\langle S \rangle\rangle = \langle\langle S_1 \rangle\rangle \vee \langle\langle S_2 \rangle\rangle \vee \cdots \vee \langle\langle S_k \rangle\rangle$  and  $\langle\langle S \rangle\rangle$  is sji, it follows that  $\langle\langle S \rangle\rangle = \langle\langle S_j \rangle\rangle$  for some  $j$ , whence  $S$  is not a minimal order generator.

If a finite semigroup  $S$  is  $\times$ -prime (e.g.  $\text{KN}$ ), then  $S$  is a minimal order generator and any minimal order generator for  $\langle\langle S \rangle\rangle$  is isomorphic to  $S$ . The proof is clear.

However, minimal order generators for the same compact pseudovariety need not be isomorphic; for example,  $\langle\langle Q_8 \rangle\rangle = \langle\langle D_4 \rangle\rangle$  and  $Q_8 \not\cong D_4$ . Since the pseudovariety  $\langle\langle Q_8 \rangle\rangle = \langle\langle D_4 \rangle\rangle$  is conjectured to be  $\text{ji}$ , it appears that minimal order generators for  $\text{ji}$  pseudovarieties need not be isomorphic.

It should be pointed out that a finite semigroup  $S$  being  $\text{sdi}$  does not imply that the pseudovariety  $\langle\langle S \rangle\rangle$  is  $\text{ji}$  or even  $\text{sji}$ . For example, the Rees matrix semigroup  $S = \mathcal{M}^0(\mathbb{Z}_2, \{1, 2\}, \{1, 2\}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix})$  is  $\text{sdi}$ , but  $\langle\langle S \rangle\rangle = \langle\langle B_2 \rangle\rangle \vee \langle\langle \mathbb{Z}_2 \rangle\rangle$  is not  $\text{sji}$ ; see [20, Section 4.7].

**1.7. Organization.** The text is organized as follows. In Section 2, the operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  is introduced in detail and some related results are established. In Section 3, some important small semigroups that are required for the present article are defined. In Section 4, some general results regarding  $\text{ji}$  pseudovarieties are established. In Section 5, some explicit pseudovarieties are shown to be  $\text{ji}$ , and conditions sufficient for a finite semigroup to generate one of them are established. In Section 6, some conditions sufficient for a finite semigroup to generate a non- $\text{ji}$  pseudovariety are established. Results in Sections 4–6 are then employed in Section 7 to prove that among all pseudovarieties generated by a semigroup of order up to five, only 30 are  $\text{ji}$ .

## 2. AUGMENTED SEMIGROUPS

All semigroups and transformation semigroups, with the exception of free semigroups and free profinite semigroups, are assumed finite. Notation in the monograph [20] will often be followed closely.

Let  $(X, S)$  be a transformation semigroup where  $S$  is a semigroup that acts faithfully on the right of a set  $X$ . Then  $\overline{(X, S)} = (X, S \cup \overline{X})$  where  $\overline{X}$  is the set of constant maps on  $X$ . The constant map to a fixed element  $x \in X$  is denoted by  $\overline{x}$ . If  $(X, S)$  and  $(Y, T)$  are transformation semigroups, then

$$(X, S) \times (Y, T) = (X \times Y, S \times T)$$

with the action  $(x, y)(s, t) = (xs, yt)$ .

Refer to Eilenberg [3] for the definition of division  $\prec$  of transformation semigroups.

**Lemma 2.1** (Eilenberg [3, Exercise I.4.1, Propositions I.5.4, and page 20]). *Let  $(X, S)$  and  $(Y, T)$  be any transformation semigroups. Then*

- (i)  $(X, S) \prec (Y, T)$  implies that  $\overline{(X, S)} \prec \overline{(Y, T)}$ ;
- (ii)  $\overline{(X, S)} \prec \overline{(Y, T)}$  implies that  $S \prec T$ ;
- (iii)  $\overline{(X, S)} \times \overline{(Y, T)} \prec \overline{(X, S)} \times \overline{(Y, T)}$ .

Lemma 2.1(ii) holds because the mappings involved are total.

**Lemma 2.2** (D. Allen; see Eilenberg [3, Proposition I.9.8]). *If  $(X, S)$  is any transformation semigroup, then  $(S^\bullet, S) \prec (X, S)^{|X|}$ .*

Following [20, Chapter 4], write  $\overline{(S^\bullet, S)} = (S^\bullet, S^{\text{bar}})$  and call  $S^{\text{bar}}$  the *augmentation* of  $S$ . Note that if  $(X, S) \prec (S^\bullet, S)$ , then  $(X, S) \prec (S^\bullet, S) \prec$

$(X, S)^{|X|}$  by Lemma 2.2 and hence

$$\overline{(X, S)} \prec \overline{(S^\bullet, S)} \prec \overline{(X, S)}^{|X|}.$$

Thus if  $S' = S \cup \overline{X}$ , then  $S' \prec S^{\text{bar}} \prec (S')^{|X|}$ , yielding the following result.

**Corollary 2.3.** *If  $(X, S)$  is a transformation semigroup such that  $(X, S) \prec (S^\bullet, S)$ , then  $\langle\langle S \cup \overline{X} \rangle\rangle = \langle\langle S^{\text{bar}} \rangle\rangle$ . In particular, if  $S$  is any semigroup and  $J$  is any right ideal of  $S$  on which it acts faithfully, then  $\langle\langle S^{\text{bar}} \rangle\rangle = \langle\langle S \cup \overline{J} \rangle\rangle$ .*

The following are some elementary properties enjoyed by augmentation.

**Proposition 2.4.** *Let  $S$  and  $T$  be any finite semigroups. Then*

- (i)  $S \prec T$  implies that  $S^{\text{bar}} \prec T^{\text{bar}}$ ;
- (ii)  $(S \times T)^{\text{bar}} \prec S^{\text{bar}} \times T^{\text{bar}}$ .

*Proof.* (i) Suppose that  $S \prec T$ , so that  $(S^\bullet, S) \prec (T^\bullet, T)$  by Eilenberg [3, Proposition I.5.8]. Then by Lemma 2.1(i),

$$(S^\bullet, S^{\text{bar}}) = \overline{(S^\bullet, S)} \prec \overline{(T^\bullet, T)} = (T^\bullet, T^{\text{bar}}).$$

Therefore  $S^{\text{bar}} \prec T^{\text{bar}}$  by Lemma 2.1(ii).

- (ii) First note that  $((S \times T)^\bullet, S \times T) \prec (S^\bullet \times T^\bullet, S \times T)$ . Then

$$\begin{aligned} & ((S \times T)^\bullet, (S \times T)^{\text{bar}}) \\ &= \overline{((S \times T)^\bullet, S \times T)} \prec \overline{(S^\bullet \times T^\bullet, S \times T)} && \text{by Lemma 2.1(i)} \\ &= \overline{(S^\bullet, S)} \times \overline{(T^\bullet, T)} \prec \overline{(S^\bullet, S)} \times \overline{(T^\bullet, T)} && \text{by Lemma 2.1(iii)} \\ &= (S^\bullet, S^{\text{bar}}) \times (T^\bullet, T^{\text{bar}}) = (S^\bullet \times T^\bullet, S^{\text{bar}} \times T^{\text{bar}}). \end{aligned}$$

Therefore  $(S \times T)^{\text{bar}} \prec S^{\text{bar}} \times T^{\text{bar}}$  by Lemma 2.1(ii).  $\square$

In the following, augmentation is viewed as a continuous operator on the lattice  $\mathbf{PV}$  of pseudovarieties. An operator is *continuous* if it preserves order and directed joins [20]. For any pseudovariety  $\mathbf{V}$ , define

$$\mathbf{V}^{\text{bar}} = \langle\langle S^{\text{bar}} \mid S \in \mathbf{V} \rangle\rangle.$$

Let  $\mathbf{RZ}$  denote the pseudovariety of right zero semigroups.

**Proposition 2.5.** *The operator on  $\mathbf{PV}$  defined by  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  is continuous, non-decreasing, and idempotent. Further,*

- (i)  $\langle\langle S \rangle\rangle^{\text{bar}} = \langle\langle S^{\text{bar}} \rangle\rangle$  for any finite semigroup  $S$ ;
- (ii)  $\mathbf{RZ} \subseteq \mathbf{V}^{\text{bar}}$  for any nontrivial pseudovariety  $\mathbf{V}$ .

*Consequently, if  $\langle\langle S \rangle\rangle = \langle\langle T \rangle\rangle$ , then  $\langle\langle S^{\text{bar}} \rangle\rangle = \langle\langle T^{\text{bar}} \rangle\rangle$ .*

*Proof.* Clearly augmentation is order preserving. Let  $\{\mathbf{V}_\delta \mid \delta \in D\}$  be any directed set of pseudovarieties, so that the complete join  $\mathbf{V} = \bigvee_{\delta \in D} \mathbf{V}_\delta$  is a union. The inclusion  $\mathbf{V}_\delta^{\text{bar}} \subseteq \mathbf{V}^{\text{bar}}$  clearly holds for all  $\delta \in D$ , so that the inclusion  $\bigvee_{\delta \in D} \mathbf{V}_\delta^{\text{bar}} \subseteq \mathbf{V}^{\text{bar}}$  also holds. Conversely, if  $S \in \mathbf{V}^{\text{bar}}$ , say  $S \prec T_1^{\text{bar}} \times T_2^{\text{bar}} \times \cdots \times T_k^{\text{bar}}$  for some  $T_1, T_2, \dots, T_k \in \mathbf{V}$ , then due to directedness, there exists  $\delta \in D$  with  $T_1, T_2, \dots, T_k \in \mathbf{V}_\delta$ , whence  $S \in \mathbf{V}_\delta^{\text{bar}}$ . Therefore augmentation is continuous.

Since  $S \prec S^{\text{bar}}$ , it is obvious that augmentation is non-decreasing and the inclusion  $\mathbf{V}^{\text{bar}} \subseteq (\mathbf{V}^{\text{bar}})^{\text{bar}}$  holds. To establish the reverse inclusion,

it suffices to prove that  $(S^{\text{bar}})^{\text{bar}} \in \mathbf{V}^{\text{bar}}$  for all  $S \in \mathbf{V}$ . But  $S^{\text{bar}}$  acts faithfully on the right of its minimal ideal  $\overline{S^\bullet}$  and it contains all the constant mappings. Thus  $(\overline{S^\bullet}, S^{\text{bar}}) = (\overline{S^\bullet}, S^{\text{bar}})$  and  $(\overline{S^\bullet}, S^{\text{bar}}) \prec ((S^{\text{bar}})^\bullet, S^{\text{bar}})$ . It follows from Corollary 2.3 that  $S^{\text{bar}} = S^{\text{bar}} \cup \overline{S^\bullet}$  generates the same pseudovariety as  $(S^{\text{bar}})^{\text{bar}}$ . This shows that  $(S^{\text{bar}})^{\text{bar}} \in \mathbf{V}^{\text{bar}}$ , so that augmentation is idempotent.

It remains to establish parts (i) and (ii).

(i) The inclusion  $\langle\langle S^{\text{bar}} \rangle\rangle \subseteq \langle\langle S \rangle\rangle^{\text{bar}}$  holds trivially. To establish the reverse inclusion, suppose that  $T \in \langle\langle S \rangle\rangle^{\text{bar}}$ , so that  $T \prec U^{\text{bar}}$  for some  $U \in \langle\langle S \rangle\rangle$ . Then  $U \prec S^n$  for some  $n \geq 0$  and so  $T \prec U^{\text{bar}} \prec (S^n)^{\text{bar}} \prec (S^{\text{bar}})^n$  by Proposition 2.4(ii). Therefore  $T \in \langle\langle S^{\text{bar}} \rangle\rangle$ . Consequently,  $\langle\langle S^{\text{bar}} \rangle\rangle = \langle\langle S \rangle\rangle^{\text{bar}}$ .

(ii) If  $S$  is a nontrivial semigroup in  $\mathbf{V}$ , then the right zero semigroup  $R_2$  of order two is a subsemigroup of  $S^{\text{bar}}$ , whence  $\mathbf{RZ} \subseteq \mathbf{V}$ .  $\square$

**Corollary 2.6.** *Suppose that  $S$  is any finite semigroup whose minimal ideal  $J$  consists of right zeroes such that  $S$  acts faithfully on the right of  $J$ . Then  $\langle\langle S \rangle\rangle^{\text{bar}} = \langle\langle S \rangle\rangle$ .*

*Proof.* By Proposition 2.5, it suffices to prove that  $\langle\langle S^{\text{bar}} \rangle\rangle = \langle\langle S \rangle\rangle$ . But since  $(J, S) = (J, S)$ , it follows that  $S = S \cup \overline{J}$ . The desired conclusion then follows from Corollary 2.3.  $\square$

### 3. SOME IMPORTANT SEMIGROUPS

In this section, semigroups that are required throughout the article are introduced. Semigroups are given by their presentations, and whenever feasible, multiplication tables. In presentations, the symbols  $\mathbf{e}$  and  $\mathbf{f}$  are exclusively reserved for idempotent elements.

**3.1. Cyclic groups.** The cyclic group of order  $n \geq 1$  is

$$\mathbb{Z}_n = \langle \mathbf{g} \mid \mathbf{g}^n = 1 \rangle = \{1, \mathbf{g}, \mathbf{g}^2, \dots, \mathbf{g}^{n-1}\}.$$

The augmentation of  $\mathbb{Z}_2 = \{1, \mathbf{g}\}$  is the semigroup  $\mathbb{Z}_2^{\text{bar}} = \{1, \mathbf{g}, \overline{1}, \overline{\mathbf{g}}\}$  given by the following multiplication table:

$\mathbb{Z}_2^{\text{bar}}$	1	$\mathbf{g}$	$\overline{1}$	$\overline{\mathbf{g}}$
1	1	$\mathbf{g}$	$\overline{1}$	$\overline{\mathbf{g}}$
$\mathbf{g}$	$\mathbf{g}$	1	$\overline{1}$	$\overline{\mathbf{g}}$
$\overline{1}$	$\overline{1}$	$\overline{\mathbf{g}}$	$\overline{1}$	$\overline{\mathbf{g}}$
$\overline{\mathbf{g}}$	$\overline{\mathbf{g}}$	$\overline{1}$	$\overline{1}$	$\overline{\mathbf{g}}$

Information on identities satisfied by the semigroups  $\mathbb{Z}_n$  and  $\mathbb{Z}_2^{\text{bar}}$  is given in Subsections 5.2 and 5.3, respectively.

**3.2. Nilpotent semigroups.** The monogenic nilpotent semigroup of order  $n \geq 1$  is

$$N_n = \langle \mathbf{a} \mid \mathbf{a}^n = 0 \rangle = \{0, \mathbf{a}, \mathbf{a}^2, \dots, \mathbf{a}^{n-1}\}.$$

The augmentation of  $N_2 = \{0, a\}$  is the semigroup  $N_2^{\text{bar}} = \{\bar{0}, a, \bar{a}, \bar{I}\}$  given by the following multiplication table:

$N_2^{\text{bar}}$	$\bar{0}$	$a$	$\bar{a}$	$\bar{I}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{I}$
$a$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{I}$
$\bar{a}$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{I}$
$\bar{I}$	$\bar{0}$	$\bar{a}$	$\bar{a}$	$\bar{I}$

Information on identities satisfied by the semigroups  $N_n$ ,  $N_n^I$ ,  $N_2^{\text{bar}}$ , and  $(N_2^{\text{bar}})^I$  is given in Subsections 5.4, 5.5, 5.6, and 5.7, respectively.

**3.3. Bands.** The smallest nontrivial bands are the semilattice  $Sl_2 = \{0, 1\}$ , the left zero semigroup

$$L_2 = \langle e, f \mid e^2 = ef = e, f^2 = fe = f \rangle = \{e, f\},$$

and the right zero semigroup

$$R_2 = \langle e, f \mid e^2 = fe = e, f^2 = ef = f \rangle = \{e, f\}.$$

These semigroups can also be given by the following multiplication tables:

$Sl_2$	0	1	$L_2$	e	f	$R_2$	e	f
0	0	0	e	e	e	e	e	f
1	0	1	f	f	f	f	e	f

Note that  $Sl_2 \cong N_1^I$  and  $L_2^{\text{op}} \cong R_2$ . It is well known that  $Sl_2$  generates the pseudovariety **Sl** of semilattices,  $L_2$  generates the pseudovariety **LZ** of left zero semigroups, and  $R_2$  generates the pseudovariety **RZ** of right zero semigroups.

The augmentation of  $L_2$  is the semigroup  $L_2^{\text{bar}} = \{e, f, \bar{e}, \bar{f}, \bar{I}\}$  given by the following multiplication table:

$L_2^{\text{bar}}$	e	f	$\bar{e}$	$\bar{f}$	$\bar{I}$
e	e	e	$\bar{e}$	$\bar{f}$	$\bar{I}$
f	f	f	$\bar{e}$	$\bar{f}$	$\bar{I}$
$\bar{e}$	$\bar{e}$	$\bar{e}$	$\bar{e}$	$\bar{f}$	$\bar{I}$
$\bar{f}$	$\bar{f}$	$\bar{f}$	$\bar{e}$	$\bar{f}$	$\bar{I}$
$\bar{I}$	$\bar{e}$	$\bar{f}$	$\bar{e}$	$\bar{f}$	$\bar{I}$

Information on identities satisfied by the semigroups  $L_2$ ,  $L_2^I$ , and  $L_2^{\text{bar}}$  is given in Subsections 5.8, 5.9, and 5.10, respectively.

**3.4. Completely 0-simple semigroups.** The smallest completely 0-simple semigroups with zero divisors are the idempotent-generated semigroup

$$A_2 = \langle a, e \mid a^2 = 0, aea = a, e^2 = eae = e \rangle = \{0, a, e, ae, ea\}$$

and the Brandt semigroup

$$B_2 = \langle a, b \mid a^2 = b^2 = 0, aba = a, bab = b \rangle = \{0, a, b, ab, ba\}.$$

Alternately,  $A_2$  and  $B_2$  can be given as the Rees matrix semigroups

$$A_2 = \mathcal{M}^0(\{1\}, \{1, 2\}, \{1, 2\}; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \quad \text{and} \quad B_2 = \mathcal{M}^0(\{1\}, \{1, 2\}, \{1, 2\}; \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}).$$

These semigroups contain subsemigroups isomorphic to

$$A_0 = \langle e, f \mid e^2 = e, f^2 = f, ef = 0 \rangle = \{0, e, f, fe\};$$

$$B_0 = \langle a, e, f \mid e^2 = e, f^2 = f, ef = fe = 0, ea = af = a \rangle = \{0, a, e, f\};$$

$$\ell_3 = \langle a, e \mid ae = 0, ea = a, e^2 = e \rangle = \{0, a, e\}.$$

All these semigroups can also be given by the following multiplication tables:

$A_2$	0	a	ae	ea	e
0	0	0	0	0	0
a	0	0	0	a	ae
ae	0	a	ae	a	ae
ea	0	0	0	ea	e
e	0	ea	e	ea	e

$B_2$	0	a	ab	ba	b
0	0	0	0	0	0
a	0	0	0	a	ab
ab	0	a	ab	0	0
ba	0	0	0	ba	b
b	0	ba	b	0	0

$A_0$	0	fe	f	e
0	0	0	0	0
fe	0	0	0	fe
f	0	fe	f	fe
e	0	0	0	e

$B_0$	0	a	e	f
0	0	0	0	0
a	0	0	0	a
e	0	a	e	0
f	0	0	0	f

$\ell_3$	0	a	e
0	0	0	0
a	0	0	0
e	0	a	e

Note that  $A_0 \cong A_2 \setminus \{e\}$ ,  $B_0 \cong B_2 \setminus \{b\}$ , and  $\ell_3 \cong A_0 \setminus \{e\} \cong B_0 \setminus \{f\}$ .

The augmentation of  $\ell_3$  is the semigroup  $\ell_3^{\text{bar}} = \{\bar{0}, a, e, \bar{a}, \bar{e}\}$  given by the following multiplication table:

$\ell_3^{\text{bar}}$	$\bar{0}$	a	e	$\bar{a}$	$\bar{e}$
$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{e}$
a	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{e}$
e	$\bar{0}$	a	e	$\bar{a}$	$\bar{e}$
$\bar{a}$	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{a}$	$\bar{e}$
$\bar{e}$	$\bar{0}$	$\bar{a}$	$\bar{e}$	$\bar{a}$	$\bar{e}$

Information on identities satisfied by the semigroups  $A_0$ ,  $A_0^I$ ,  $A_2$ ,  $B_2$ , and  $\ell_3^{\text{bar}}$  is given in Subsections 5.11, 5.12, 5.13, 5.14, and 5.15, respectively.

The semigroup  $\ell_3$  belongs to an infinite class of semigroups  $S$  with the property that  $\langle\langle S \rangle\rangle$  is not ji but  $\langle\langle S \rangle\rangle^{\text{bar}}$  is ji; see Subsection 4.5. The semigroup  $B_0$  serves as a counterexample to the implications

$$S \text{ is sji} \implies S \text{ is ji} \quad \text{and} \quad S \text{ is sji} \implies S^I \text{ is sji}$$

mentioned in the introduction.

**Proposition 3.1.** (i) *The pseudovariety  $\langle\langle B_0 \rangle\rangle$  is sji.*

(ii) *The pseudovariety  $\langle\langle B_0 \rangle\rangle$  is not ji.*

(iii) *The pseudovariety  $\langle\langle B_0^I \rangle\rangle$  is not sji.*

*Proof.* The pseudovariety  $\langle\langle B_0 \rangle\rangle$  is **sj**i because it has a unique maximal proper subpseudovariety [6, Lemma 5(b)]. The pseudovariety  $\langle\langle B_0^I \rangle\rangle$  is not **sj**i because it has two maximal proper subpseudovarieties [6, Lemma 6(b)]. If the pseudovariety  $\langle\langle B_0 \rangle\rangle$  is **ji**, then the pseudovariety  $\langle\langle B_0^I \rangle\rangle$  is also **ji** (see Lemma 5.2), whence  $\langle\langle B_0^I \rangle\rangle$  is contradictorily **sj**i.  $\square$

#### 4. SOME GENERAL RESULTS ON JOIN IRREDUCIBILITY

The pseudovariety defined by a class  $\Sigma$  of pseudoidentities is denoted by  $[\Sigma]$ , while the pseudovariety generated by a class  $\mathcal{K}$  of finite semigroups is denoted by  $\langle\langle \mathcal{K} \rangle\rangle$ . A pseudovariety is *compact* if it is generated by a single finite semigroup. The *exclusion class*  $\text{Excl}(S)$  of a finite semigroup  $S$  is the class of all finite semigroups  $T$  for which  $S \notin \langle\langle T \rangle\rangle$ . Recall that a finite semigroup  $S$  is **ji** if and only if  $\text{Excl}(S)$  is a pseudovariety [20, Theorem 7.1.2].

In the present section, some results on the property of being **ji** are established. There are six subsections. The main result of Subsection 4.1 demonstrates that many exclusion classes of **ji** semigroups in the present article are not definable by a certain type of pseudoidentities. In Subsection 4.2, the notion of a “large” pseudovariety is introduced. It turns out that the exclusion class of a **ji** semigroup that is right letter mapping, left letter mapping, or group mapping satisfies this largeness condition. In Subsection 4.3, it is shown that the operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  on  $\mathbf{PV}$  preserves the property of being **ji**. More specifically, if  $\mathbf{u} \approx \mathbf{v}$  is a pseudoidentity that defines the exclusion class  $\text{Excl}(S)$  of a **ji** semigroup  $S$ , then it is shown how a pseudoidentity that defines  $\text{Excl}(S^{\text{bar}})$  can be obtained from  $\mathbf{u} \approx \mathbf{v}$ .

In Subsection 4.4, it is shown that alternately performing the operators  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  and  $\mathbf{V} \mapsto \mathbf{V}^{\flat} = \langle\langle ((S^{\text{op}})^{\text{bar}})^{\text{op}} \mid S \in \mathbf{V} \rangle\rangle$  on a nontrivial pseudovariety  $\langle\langle S \rangle\rangle$  results in an infinite increasing chain of pseudovarieties; if the semigroup  $S$  is **ji** to begin with, then the pseudovarieties are all **ji**. In Subsection 4.5, an infinite class  $\{O_k \mid k \geq 2\}$  of finite semigroups is introduced and shown to satisfy the following property: for each  $k \geq 2$ , the pseudovariety  $\langle\langle O_k \rangle\rangle$  is not **ji**, while the pseudovariety  $\langle\langle O_k \rangle\rangle^{\text{bar}}$  is **ji**.

In Subsection 4.6, a sufficient condition, due to G. M. Bergman, is presented under which a finite **sdi** group is **ji**.

**4.1. Non-definability by simple pseudoidentities.** For this subsection, the assumption that all semigroups are finite is temporarily abandoned. The free profinite semigroup on a set  $A$  is denoted by  $\widehat{A}^+$ . A pseudoidentity  $\mathbf{u} \approx \mathbf{v}$  is *simple* if  $\mathbf{u}$  and  $\mathbf{v}$  belong to the smallest subsemigroup  $F(A)$  of  $\widehat{A}^+$  containing  $A$  that is closed under product and unary implicit operations; the latter condition means that  $\overline{\{\mathbf{w}\}}^+ \subseteq F(A)$  for all  $\mathbf{w} \in F(A)$ .

The following theorem was essentially proved by Almeida and Volkov [2], based on an earlier variant of Rhodes [19].

**Theorem 4.1.** *Suppose that  $\mathbf{V}$  is any proper pseudovariety of semigroups containing all semigroups with abelian maximal subgroups. Then  $\mathbf{V}$  cannot be defined by simple pseudoidentities.*

*Proof.* Let  $A$  be a fixed countably infinite set and for any  $m, n \geq 1$ , let  $\mathcal{B}_{m,n}$  be the variety of semigroups defined by the identity  $x^m \approx x^{m+n}$ . Then the

free semigroup  $B(1, m, n)$  on one-generator in  $\mathcal{B}_{m,n}$  is finite and if  $x^\eta \in \widehat{\{x\}^+}$ , then there exists an integer  $n_\eta \leq m+n-1$  such that  $x^\eta = x^{n_\eta}$  in  $B(1, m, n)$ . Thus each implicit operation in  $F(A)$  has a natural interpretation on any semigroup in  $\mathcal{B}(m, n)$  which agrees with its usual interpretation in finite semigroups (namely interpret  $w^\eta$  as  $w^{n_\eta}$  for every element  $w$  of a semigroup  $S \in \mathcal{B}_{m,n}$ ).

Suppose that  $\mathbf{V}$  is defined by a set  $\Sigma$  of simple pseudoidentities. Let  $\mathcal{W}$  be the variety of universal algebras defined by  $\Sigma$  in the signature  $\tau$  consisting of multiplication and all unary implicit operations and let  $T$  be a finite semigroup. Then there exist  $m \geq 6$  and  $n \geq 1$  such that  $T$  belongs to  $\mathcal{B}(m, n)$ . As discussed above,  $\mathcal{B}(m, n)$  can be viewed as a variety in the signature  $\tau$  such that the unary implicit operations have their usual interpretations in all finite semigroups in  $\mathcal{B}(m, n)$ .

Now McCammond [15] has shown that for each integer  $k \geq 1$ , the semigroup  $B(k, m, n)$  has cyclic maximal subgroups and that there is a system of cofinite ideals for  $B(k, m, n)$  with empty intersection. Therefore,  $B(k, m, n)$  is an infinite subdirect product of finite semigroups with abelian maximal subgroups. Since  $\mathcal{W}$  contains all finite semigroups with abelian maximal subgroups, it follows that  $B(k, m, n) \in \mathcal{W}$ , whence  $\mathcal{B}(m, n) \subseteq \mathcal{W}$ . Therefore  $T$  belongs to  $\mathcal{W}$  and so satisfies the pseudoidentities  $\Sigma$ . Consequently,  $T \in \mathbf{V}$  and hence  $\mathbf{V}$  is the pseudovariety of all finite semigroups.  $\square$

In the present article, pseudoidentities involving idempotents from the minimal ideal of a free profinite semigroup are often used to define the exclusion pseudovarieties of ji semigroups. Since many of these exclusion pseudovarieties contain all semigroups with abelian maximal subgroups, Theorem 4.1 implies that in general, simple pseudoidentities cannot be used in their definition. It is presently unknown if one must use idempotents from the minimal ideal.

**4.2. Large exclusion pseudovarieties.** If  $\mathbf{V}$  and  $\mathbf{W}$  are pseudovarieties of semigroups, then their *Mal'cev product*  $\mathbf{V} \mathbin{\boxtimes} \mathbf{W}$  is the pseudovariety generated by all semigroups  $S$  with a homomorphism  $\varphi: S \rightarrow T$  such that  $T \in \mathbf{W}$  and  $e\varphi^{-1} \in \mathbf{V}$  for all idempotents  $e \in T$ . We denote by  $\mathbf{1}$  the trivial pseudovariety (generated by the trivial semigroup). A remarkable property of the Mal'cev product is that

$$\left( \bigcap_{\alpha \in A} \mathbf{V}_\alpha \right) \mathbin{\boxtimes} \mathbf{W} = \bigcap_{\alpha \in A} (\mathbf{V}_\alpha \mathbin{\boxtimes} \mathbf{W}). \quad (4.1)$$

See [20] for details.

Let  $S$  be a ji semigroup. We say that  $\text{Excl}(S)$  is *large* if

$$\mathbf{1} \mathbin{\boxtimes} \text{Excl}(S) = \text{Excl}(S).$$

If  $\text{Excl}(S)$  is large and if  $\{\mathbf{V}_\alpha\}_{\alpha \in A}$  is a collection of pseudovarieties with trivial intersection  $\bigcap_{\alpha \in A} \mathbf{V}_\alpha = \mathbf{1}$ , then it follows from (4.1) and the fact that  $\text{Excl}(S)$  is mi that  $\mathbf{V}_\alpha \mathbin{\boxtimes} \text{Excl}(S) = \text{Excl}(S)$  for some  $\alpha \in A$ . In particular, either  $\mathbf{A} \mathbin{\boxtimes} \text{Excl}(S) = \text{Excl}(S)$  or  $\mathbf{G} \mathbin{\boxtimes} \text{Excl}(S) = \text{Excl}(S)$  where  $\mathbf{A}$  is the pseudovariety of aperiodic semigroups and  $\mathbf{G}$  is the pseudovariety of groups. For more examples of pseudovarieties with trivial intersection, see [20].



If  $S$  is a finite subdirectly indecomposable semigroup, then  $S$  has a unique 0-minimal ideal  $I$  (where if  $S$  has no zero, then we consider the minimal ideal as 0-minimal). Moreover, one of the following cases holds:

- $I^2 = 0$  (the null case);
- $S$  acts faithfully on the right of the set of  $\mathcal{L}$ -classes of  $I$  (the left letter mapping case);
- $S$  acts faithfully on the left of the set of  $\mathcal{L}$ -classes of  $I$  (the right letter mapping case);
- $I$  contains a nontrivial maximal subgroup and  $S$  acts faithfully on both the left and right of  $I$  (the group mapping case).

In the last three cases we say that  $S$  is of *semisimple* type. See [20, Chapter 4.7].

**Theorem 4.2.** *Let  $S$  be a subdirectly indecomposable ji semigroup of semisimple type (left letter mapping, right letter mapping, or group mapping). Then  $\text{Excl}(S)$  is large.*

*Proof.* Obviously,  $\text{Excl}(S) \subseteq \mathbf{1} \boxplus \text{Excl}(S)$ . As  $\text{Excl}(S)$  is the largest pseudovariety that fails to contain  $S$ , it suffices to show that  $S \notin \mathbf{1} \boxplus \text{Excl}(S)$ . But [20, Theorem 4.6.50] immediately implies that in any of the three cases,  $S \in \mathbf{1} \boxplus \mathbf{V}$  if and only if  $S \in \mathbf{V}$  for any pseudovariety  $\mathbf{V}$ . Thus  $S \notin \mathbf{1} \boxplus \text{Excl}(S)$  and so  $\mathbf{1} \boxplus \text{Excl}(S) = \text{Excl}(S)$ .  $\square$

The proof of the above theorem is in fact valid if  $S$  is left letter mapping, right letter mapping, or group mapping even if it is not subdirectly indecomposable.

**4.3. Augmentation preserves join irreducibility.** In this subsection, augmentation is shown to preserve join irreducibility. Some special cases were previously considered in [20, Section 7.3].

**Theorem 4.3.** *The operator  $\mathbf{V} \mapsto \mathbf{V}^{\text{bar}}$  preserves the property of being ji. In particular, if a pseudovariety  $\langle\langle S \rangle\rangle$  is ji, then the pseudovariety  $\langle\langle S^{\text{bar}} \rangle\rangle$  is also ji. Further, if  $\text{Excl}(S) = [\mathbf{u} \approx \mathbf{v}]$  where  $\mathbf{u}, \mathbf{v} \in \widehat{A}^+$ , then*

$$\text{Excl}(S^{\text{bar}}) = [(\mathbf{e}z\mathbf{u})^\omega \approx (\mathbf{e}z\mathbf{v})^\omega]$$

where  $z \notin A$  and  $\mathbf{e}$  is an idempotent in the minimal ideal of  $(A \cup \{z\})^+$ .

*Proof.* First note that since  $S \notin \text{Excl}(S)$ , there exists some homomorphism  $\varphi: \widehat{A}^+ \rightarrow S$  such that  $\mathbf{u}\varphi \neq \mathbf{v}\varphi$ . Let  $1$  denote the identity element of  $S^\bullet$ , and extend  $\varphi$  to a homomorphism  $(A \cup \{z\})^+ \rightarrow S^{\text{bar}}$  by sending  $z$  to  $\bar{1}$ . Then  $(\mathbf{e}z\mathbf{u})^\omega \varphi = \bar{\mathbf{u}}\bar{\varphi} \neq \bar{\mathbf{v}}\bar{\varphi} = (\mathbf{e}z\mathbf{v})^\omega \varphi$  and so  $S^{\text{bar}} \notin [(\mathbf{e}z\mathbf{u})^\omega \approx (\mathbf{e}z\mathbf{v})^\omega]$ .

To complete the proof, it suffices to assume that  $T \notin [(\mathbf{e}z\mathbf{u})^\omega \approx (\mathbf{e}z\mathbf{v})^\omega]$ , and then show that  $S^{\text{bar}} \in \langle\langle T \rangle\rangle$ . Replacing  $T$  by a subsemigroup if necessary, generality is not lost by assuming the existence of a surjective homomorphism  $\psi: (A \cup \{z\})^+ \rightarrow T$  such that  $(\mathbf{e}z\mathbf{u})^\omega \psi \neq (\mathbf{e}z\mathbf{v})^\omega \psi$ . Now  $T$  acts on the right of the set  $B$  of  $\mathcal{L}$ -classes of its minimal ideal  $J$ ; let  $(B, \text{RLM}(T))$  denote the resulting faithful transformation semigroup. Note that  $(B, \text{RLM}(T)) = \overline{(B, \text{RLM}(T))}$  because if  $b \in B$ , then any element of  $T$  in the  $\mathcal{L}$ -class of  $b$  acts on  $B$  as a constant map to  $b$  by the structure of completely simple semigroups. It follows from Corollary 2.6 that

$\langle\langle \text{RLM}(T) \rangle\rangle^{\text{bar}} = \langle\langle \text{RLM}(T) \rangle\rangle$ , since the constant mappings form the minimal ideal of  $\text{RLM}(T)$ .

Since  $(\mathbf{e}z)\psi$  is in the minimal ideal  $J$  of  $T$ , the elements  $((\mathbf{e}z)\psi)(\mathbf{u}\psi)$  and  $((\mathbf{e}z)\psi)(\mathbf{v}\psi)$  are  $\mathcal{H}$ -equivalent. However, they are not  $\mathcal{L}$ -equivalent because otherwise they would be  $\mathcal{H}$ -equivalent and hence have the same idempotent power, as  $J$  is completely simple. Thus  $\mathbf{u}\psi$  and  $\mathbf{v}\psi$  have distinct images under the quotient map  $T \rightarrow \text{RLM}(T)$ . Consequently, there is a homomorphism  $\varphi: \hat{A}^+ \rightarrow \text{RLM}(T)$  such that  $\mathbf{u}\varphi \neq \mathbf{v}\varphi$ , that is,  $\text{RLM}(T) \notin \text{Excl}(S)$ . Therefore  $S \in \langle\langle \text{RLM}(T) \rangle\rangle$ , whence  $S^{\text{bar}} \in \langle\langle \text{RLM}(T) \rangle\rangle^{\text{bar}} = \langle\langle \text{RLM}(T) \rangle\rangle \subseteq \langle\langle T \rangle\rangle$  as required.  $\square$

**Corollary 4.4.** *Suppose that  $(X, S)$  is any transformation semigroup such that  $(X, S) \prec (S^\bullet, S)$  and that the pseudovariety  $\langle\langle S \rangle\rangle$  is ji. Then the pseudovariety  $\langle\langle S \cup \bar{X} \rangle\rangle$  is also ji.*

*Proof.* This follows from Corollary 2.3 and Theorem 4.3.  $\square$

Note that if  $S$  is ji, then  $\text{Excl}(S^{\text{bar}})$  will be large by Theorem 4.2 (and the remark following it).

#### 4.4. Iterating augmentation and its dual with applications to bands.

For any semigroup  $S$ , define

$$S^b = ((S^{\text{op}})^{\text{bar}})^{\text{op}}.$$

In other words,  $S^b$  is obtained by considering the left action of  $S$  on  $S^\bullet$  and adjoining constant maps. For any pseudovariety  $\mathbf{V}$ , define

$$\mathbf{V}^b = \langle\langle S^b \mid S \in \mathbf{V} \rangle\rangle.$$

By symmetry,  $\mathbf{V} \mapsto \mathbf{V}^b$  is a continuous idempotent operator that preserves join irreducibility. See [20, Chapter 2]. Define the operators  $\alpha, \beta: \mathbf{PV} \rightarrow \mathbf{PV}$  by  $\alpha\mathbf{V} = \mathbf{V}^{\text{bar}}$  and  $\beta\mathbf{V} = \mathbf{V}^b$ . The aim of this subsection is to show that for any nontrivial finite semigroup  $S$ , the hierarchy

$$\mathbf{V}_n = (\beta\alpha)^n \langle\langle S \rangle\rangle, \quad n \geq 0 \tag{4.2}$$

is strict, as is the dual hierarchy obtained by interchanging the roles of  $\alpha$  and  $\beta$ . An important observation is that  $\beta\alpha \langle\langle S \rangle\rangle$  is a compact pseudovariety containing  $Sl_2$  that is generated by  $(S^{\text{bar}})^b$ , which is left mapping with respect to its minimal ideal. Thus it suffices to handle the case that  $\mathbf{Sl} \subseteq \langle\langle S \rangle\rangle$  and  $S$  is left mapping with respect to its minimal ideal.

**Proposition 4.5.** *Suppose that  $S$  is any finite semigroup. Then the inclusions  $S^{\text{bar}} \in \mathbf{RZ} \textcircled{\cap} (\langle\langle S \rangle\rangle \vee \mathbf{Sl})$  and  $S^b \in \mathbf{LZ} \textcircled{\cap} (\langle\langle S \rangle\rangle \vee \mathbf{Sl})$  hold.*

*Proof.* Clearly,  $S^{\text{bar}}/\overline{S^\bullet}$  divides the semigroup  $S^0$  obtained by adjoining an external zero element 0 to  $S$ . Since  $\overline{S^\bullet}$  is a right zero semigroup and  $\langle\langle S^0 \rangle\rangle \subseteq \langle\langle S \rangle\rangle \vee \mathbf{Sl}$ , the inclusion  $S^{\text{bar}} \in \mathbf{RZ} \textcircled{\cap} (\langle\langle S \rangle\rangle \vee \mathbf{Sl})$  holds. The second inclusion is dual.  $\square$

Define the operators  $\tilde{\alpha}, \tilde{\beta}: \mathbf{PV} \rightarrow \mathbf{PV}$  by  $\tilde{\alpha}\mathbf{V} = \mathbf{RZ} \textcircled{\cap} \mathbf{V}$  and  $\tilde{\beta}\mathbf{V} = \mathbf{LZ} \textcircled{\cap} \mathbf{V}$ . These operators are idempotent. For any finite semigroup  $S$  that contains  $Sl_2$  as a subsemigroup, define the hierarchy

$$\mathbf{U}_n = (\tilde{\beta}\tilde{\alpha})^n \langle\langle S \rangle\rangle, \quad n \geq 0. \tag{4.3}$$

Observe that  $\mathbf{V}_n \subseteq \mathbf{U}_n$  for all  $n \geq 0$  as a consequence of Proposition 4.5.

**Proposition 4.6.** *Suppose that  $S$  is any nontrivial band that is left mapping with respect to its minimal ideal and that  $\mathbf{V}$  is any pseudovariety such that  $\mathbf{Sl} \subseteq \mathbf{V}$ . Then  $S^{\text{bar}} \in \mathbf{RZ} \mathbin{\boxtimes} \mathbf{V}$  if and only if  $S \in \mathbf{V}$ .*

*Proof.* If  $S \in \mathbf{V}$ , then  $S^{\text{bar}} \in \mathbf{RZ} \mathbin{\boxtimes} \mathbf{V}$  by Proposition 4.5. Conversely, since  $S^{\text{bar}}$  is a band,  $S^{\text{bar}} \in \mathbf{RZ} \mathbin{\boxtimes} \mathbf{V}$  if and only if  $S^{\text{bar}} \in \mathbf{D} \mathbin{\boxtimes} \mathbf{V}$ , where  $\mathbf{D}$  is the pseudovariety of semigroups whose idempotents are right zeroes; this occurs if and only if the quotient of  $S^{\text{bar}}$  by the intersection  $\text{LM}$  of all its left mapping congruences belongs to  $\mathbf{V}$  [20, Theorem 4.6.50]. Note that since  $S$  is a left mapping band with respect to its minimal ideal, its minimal ideal consists of at least two left zeroes. Therefore the minimal ideal of  $S^{\text{bar}}$  contains no elements of  $S$ . Then  $S^{\text{bar}}/\text{LM} \cong S^0$  because  $S^{\text{bar}}$  acts trivially on the left of its minimal ideal and acts as  $S$  does on the left of its other  $\mathcal{J}$ -classes. Since  $Sl_2 \in \mathbf{V}$ , it follows that  $S^{\text{bar}}/\text{LM} \in \mathbf{V}$  if and only if  $S \in \mathbf{V}$ .  $\square$

**Corollary 4.7.** *Suppose that  $S$  is any nontrivial band that is left mapping with respect to its minimal ideal and that  $\mathbf{V}$  is any pseudovariety such that  $\mathbf{Sl} \subseteq \mathbf{V}$ . Then  $(S^{\text{bar}})^{\flat} \in \widetilde{\beta\alpha}\mathbf{V}$  if and only if  $S \in \mathbf{V}$ .*

*Proof.* Since  $S^{\text{bar}}$  is a nontrivial band that is right mapping with respect to its minimal ideal, the dual of Proposition 4.6 implies that  $(S^{\text{bar}})^{\flat} \in \mathbf{LZ} \mathbin{\boxtimes} (\mathbf{RZ} \mathbin{\boxtimes} \mathbf{V})$  if and only if  $S^{\text{bar}} \in \mathbf{RZ} \mathbin{\boxtimes} \mathbf{V}$ . An application of Proposition 4.6 then yields that  $(S^{\text{bar}})^{\flat} \in \mathbf{LZ} \mathbin{\boxtimes} (\mathbf{RZ} \mathbin{\boxtimes} \mathbf{V})$  if and only if  $S \in \mathbf{V}$ .  $\square$

The hierarchies (4.2) and (4.3) for the case  $S = Sl_2$  are now analyzed. Let  $\mathbf{B}$  denote the pseudovariety of bands.

**Lemma 4.8.** *Consider the hierarchies (4.2) and (4.3) with  $S = Sl_2$ . Then*

- (i)  $\mathbf{V}_n \not\subseteq \mathbf{U}_{n-1}$  for all  $n \geq 1$ ;
- (ii) the hierarchies (4.2) and (4.3) are strict;
- (iii)  $\bigcup_{n \geq 0} \mathbf{U}_n = \bigcup_{n \geq 0} \mathbf{V}_n = \mathbf{B}$ .

*Proof.* (i) This is established by induction on  $n$ . The exclusion  $\mathbf{V}_1 \not\subseteq \mathbf{U}_0$  holds since  $S^{\text{bar}} \in \mathbf{V}_1$  while  $S^{\text{bar}} \notin \mathbf{Sl} = \mathbf{U}_0$  due to  $R_2 \subseteq S^{\text{bar}}$ . Suppose that  $\mathbf{V}_n \not\subseteq \mathbf{U}_{n-1}$  for some  $n \geq 2$ . Note that  $\mathbf{V}_n$  is generated by a band of the form  $T = R^{\flat}$  and so  $T$  is left mapping with respect to its minimal ideal. Since  $T \notin \mathbf{U}_{n-1}$ , it follows from Corollary 4.7 that  $(T^{\text{bar}})^{\flat} \notin \mathbf{U}_n$ . Therefore  $(T^{\text{bar}})^{\flat} \in \mathbf{V}_{n+1} \setminus \mathbf{U}_n$ , whence  $\mathbf{V}_{n+1} \not\subseteq \mathbf{U}_n$ .

(ii) Since  $\mathbf{V}_n \not\subseteq \mathbf{U}_{n-1}$  by part (i) and  $\mathbf{V}_{n-1} \subseteq \mathbf{U}_{n-1}$ , the hierarchy (4.2) is strict. Similarly,  $\mathbf{V}_n \subseteq \mathbf{U}_n$  and  $\mathbf{V}_n \not\subseteq \mathbf{U}_{n-1}$  imply that the hierarchy (4.3) is strict.

(iii) This result holds because the lattice of band pseudovarieties is well known not to contain any strictly increasing infinite chain of subpseudovarieties whose union is not  $\mathbf{B}$ .  $\square$

**Theorem 4.9.** *The hierarchy (4.2) is strict for any nontrivial finite semigroup  $S$ .*

*Proof.* Since the hierarchy stabilizes as soon as two consecutive pseudovarieties are identical, replacing  $S$  by  $(S^{\text{bar}})^{\flat}$  if necessary,  $S$  can be assumed to contain  $Sl_2$  as a subsemigroup. It then follows from Lemma 4.8 that

$\bigcup_{n \geq 0} \mathbf{V}_n$  contains the pseudovariety  $\mathbf{B}$ . But since  $\mathbf{B}$  is not contained in any compact pseudovariety [22], the union  $\bigcup_{n \geq 0} \mathbf{V}_n$  is not compact. Since each  $\mathbf{V}_n$  is compact, the hierarchy is strict.  $\square$

**Corollary 4.10.** *If  $\langle\langle S \rangle\rangle$  is ji, then the pseudovarieties*

$$\langle\langle S \rangle\rangle, \langle\langle S^{\text{bar}} \rangle\rangle, \langle\langle (S^{\text{bar}})^{\flat} \rangle\rangle, \langle\langle ((S^{\text{bar}})^{\flat})^{\text{bar}} \rangle\rangle, \dots$$

*are ji; these pseudovarieties are all distinct except possibly for  $\langle\langle S \rangle\rangle = \langle\langle S^{\text{bar}} \rangle\rangle$ . A dual result holds when  $^{\flat}$  is first applied before  $^{\text{bar}}$ .*

**Corollary 4.11.** *The pseudovariety  $\mathbf{B}$  is fji.*

*Proof.* Since  $\mathbf{Sl}$  is ji, each step in the hierarchy (4.2) is ji with  $S = Sl_2$ . As the union of a chain of ji pseudovarieties is fji [20, Chapter 7], it follows from Lemma 4.8 that  $\mathbf{B}$  is fji.  $\square$

Using the known structure of the lattice of band pseudovarieties [16] (which coincides with the lattice of band varieties), we can say more. Namely, we will show that any sji band is ji. Recall that  $\mathbf{LNB} = \mathbf{Sl} \vee \mathbf{LZ}$ .

**Proposition 4.12.** *The pseudovariety  $\mathbf{RZ} \mathbin{\text{\textcircled{m}}} \mathbf{LNB}$  is generated by  $L_2^{\text{bar}}$ .*

*Proof.* It follows from Pastijn [16, Figure 3] and the description of the lattice of band pseudovarieties (see, for example, Almeida [1, Figure 5.1]) that  $\langle\langle L_2, R_2^I \rangle\rangle = \llbracket x^2 \approx x, xyz \approx xzyz \rrbracket$  is the unique maximal subpseudovariety of  $\mathbf{RZ} \mathbin{\text{\textcircled{m}}} \mathbf{LNB} = \llbracket x^2 \approx x, xyz \approx xzxyz \rrbracket$ . It is then routinely checked that  $L_2^{\text{bar}} \in \mathbf{RZ} \mathbin{\text{\textcircled{m}}} \mathbf{LNB} \setminus \langle\langle L_2, R_2^I \rangle\rangle$ . Consequently,  $\langle\langle L_2^{\text{bar}} \rangle\rangle = \mathbf{RZ} \mathbin{\text{\textcircled{m}}} \mathbf{LNB}$ .  $\square$

By Proposition 4.12 and results from Pastijn [16], a description of proper sji pseudovarieties of bands can be given as follows. Let  $S = L_2^{\text{bar}}$  and  $T = R_2^{\flat}$ . Then the proper nontrivial sji pseudovarieties of bands are  $\mathbf{LZ}$ ,  $\mathbf{RZ}$ , and those pseudovarieties that can be obtained by applying an alternating word  $W(\tilde{\alpha}, \tilde{\beta})$  over  $\{\tilde{\alpha}, \tilde{\beta}\}$  to the pseudovarieties generated by  $S$ ,  $T$ , or  $Sl_2$  (where the last letter of  $W$  should be  $\tilde{\beta}$  when starting from  $\langle\langle S \rangle\rangle$  and should be  $\tilde{\alpha}$  when starting from  $\langle\langle T \rangle\rangle$ ). Further, there are no sji pseudovarieties strictly in between any successive iterations of these operators. Since  $\alpha \mathbf{V} \leq \tilde{\alpha} \mathbf{V}$ ,  $\beta \mathbf{V} \leq \tilde{\beta} \mathbf{V}$  for any pseudovariety  $\mathbf{V}$  contain  $\mathbf{Sl}$ , and each successive iteration of  $\alpha$  and  $\beta$  starting from the pseudovariety generated by one of  $S$ ,  $T$  or  $Sl_2$  (where the rightmost operator applied must be  $\beta$  for  $S$  and  $\alpha$  for  $T$ ) results in a new ji pseudovariety, it follows that if  $W(x, y)$  is any alternating word over  $\{x, y\}$ , then  $W(\alpha, \beta) \mathbf{V} = W(\tilde{\alpha}, \tilde{\beta}) \mathbf{V}$  whenever  $\mathbf{V}$  is one of the pseudovarieties generated by  $S$ ,  $T$ , or  $Sl_2$ . Consequently, each proper sji pseudovariety of bands is, in fact, ji by Corollary 4.10. The following result is thus established.

**Theorem 4.13.** *Any sji band is ji, that is, a proper pseudovariety of bands is sji if and only if it is ji.*

In particular, since sji is a decidable property, ji is also decidable for finite bands. The answer to Question 1.2 is thus affirmative for bands.

**4.5. From non-ji pseudovarieties to ji pseudovarieties.** For each  $k \geq 2$ , define the semigroup

$$O_k = \langle x, e \mid x^k = x^{k-1}e = 0, ex = x, e^2 = e \rangle.$$

The main goal of the present subsection is to show that the pseudovariety  $\langle\langle O_k \rangle\rangle$  is not ji whereas the pseudovariety  $\langle\langle O_k \rangle\rangle^{\text{bar}}$  is ji. It is also shown that the pseudovarieties  $\langle\langle O_2 \rangle\rangle^{\text{bar}}, \langle\langle O_3 \rangle\rangle^{\text{bar}}, \langle\langle O_4 \rangle\rangle^{\text{bar}}, \dots$  are all distinct.

It is easily seen that the semigroups  $O_2$  and  $\ell_3$  are isomorphic by referring to the presentations. Since the semigroup  $\ell_3^{\text{bar}}$  is of order five (Subsection 3.4), the ji pseudovariety  $\langle\langle \ell_3^{\text{bar}} \rangle\rangle = \langle\langle O_2 \rangle\rangle^{\text{bar}}$  is required later in the article (Theorem 5.29).

**Lemma 4.14.** *For each  $k \geq 2$ , the semigroup  $O_k$  consists precisely of the following  $2k - 1$  distinct elements:*

$$0, x, x^2, \dots, x^{k-1}, e, xe, x^2e, \dots, x^{k-2}e. \quad (4.4)$$

*Proof.* It is routinely checked that (4.4) are all the elements of  $O_k$ . Therefore it remains to verify that the elements in (4.4) are distinct. Recall that the right zero semigroup of order two is  $R_2 = \{e, f\}$  and that the monogenic nilpotent semigroup of order  $k$  is

$$N_k = \langle a \mid a^k = 0 \rangle = \{0, a, a^2, \dots, a^{k-1}\}.$$

Consider the subsemigroup  $T = (N_k^I \times R_2) \setminus \{(I, e)\}$  of  $N_k^I \times R_2$  and the ideal  $J = \{(0, e), (0, f), (a^{k-1}, f)\}$  of  $T$ . Define  $\varphi: \{x, e\}^+ \rightarrow T/J$  by  $x\varphi = (a, e)$  and  $e\varphi = (I, f)$ . Then

$$x^k\varphi = (0, e) \in J, \quad (x^{k-1}e)\varphi = (a^{k-1}, f) \in J, \quad (ex)\varphi = x\varphi, \quad e^2\varphi = e\varphi.$$

It follows that  $\varphi$  induces a homomorphism  $O_k \mapsto T/J$  that separates the elements in (4.4).  $\square$

**Proposition 4.15.** *The pseudovariety  $\langle\langle O_k \rangle\rangle$  is not ji.*

*Proof.* Since  $O_k \prec T/J \prec N_k^I \times R_2$  by the proof of Lemma 4.14 (where we retain the notation of that proof), the inclusion  $\langle\langle O_k \rangle\rangle \subseteq \langle\langle N_k^I \rangle\rangle \vee \mathbf{RZ}$  holds. But  $\langle\langle N_k^I \rangle\rangle$  consists of commutative semigroups while  $\mathbf{RZ}$  consists of bands. Therefore,  $\langle\langle O_k \rangle\rangle \not\subseteq \langle\langle N_k^I \rangle\rangle$  and  $\langle\langle O_k \rangle\rangle \not\subseteq \mathbf{RZ}$ .  $\square$

It remains to prove that the pseudovariety  $\langle\langle O_k \rangle\rangle^{\text{bar}}$  is ji.

**Lemma 4.16.** *Suppose that  $U$  is any semigroup generated by two elements  $f$  and  $y$  such that  $f^2 = f$ ,  $fy = y$ , and  $y^{k-1} \notin \{y^n \mid n \geq k\}$ . Then*

- (i)  $y, y^2, \dots, y^{k-1}$  are distinct and not in  $\{y^n \mid n \geq k\}$ ;
- (ii)  $f, yf, y^2f, \dots, y^{k-2}f$  are distinct and not in  $\{y^m f \mid m \geq k-1\}$ ;
- (iii)  $y^i = y^j f$  implies that either  $i = j$  or  $i, j \geq k-1$ .

*Proof.* (i) This follows from the structure of monogenic semigroups.

(ii) Suppose that  $y^i f = y^j f$  for some  $i, j \geq 0$ . Then  $y^{i+1} = y^i f y = y^j f y = y^{j+1}$ . Therefore by part (i), either  $i = j$  or  $i, j \geq k-1$ .

(iii) Suppose that  $y^i = y^j f$  for some  $i \geq 1$  and  $j \geq 0$ . Then  $y^{i+1} = y^j f y = y^{j+1}$ . Therefore by part (i), either  $i = j$  or  $i, j \geq k-1$ .  $\square$

Recall that the inclusion  $\langle\langle O_k \rangle\rangle \subseteq \langle\langle N_k^I \rangle\rangle \vee \mathbf{RZ}$  was established in the proof of Proposition 4.15; this result is generalized in the following.

**Lemma 4.17.** *Suppose that  $T$  is any finite semigroup generated by two elements  $d$  and  $z$  such that  $d^2 = d$ ,  $dz = z$ , and  $z^{k-1} \notin \{z^n \mid n \geq k\}$ . Then  $\langle\langle O_k \rangle\rangle \subseteq \langle\langle T \rangle\rangle \vee \mathbf{RZ}$ .*

*Proof.* Consider the semigroup  $T \times R_2$  and its subsemigroup  $U = \langle y, f \rangle$  generated by  $y = (z, \mathbf{e})$  and  $f = (d, \mathbf{f})$ . Then it is routinely checked that

- (a)  $f^2 = f$ ,  $fy = y$ ,
- (b)  $y^n = (z^n, \mathbf{e})$  for all  $n \geq 1$ ,
- (c)  $y^n f = (z^n d, \mathbf{f})$  for all  $n \geq 1$ .

It follows from (b) and the assumption  $z^{k-1} \notin \{z^n \mid n \geq k\}$  that

- (d)  $y^{k-1} \notin \{y^n \mid n \geq k\}$ .

It is clear from (a) that  $U = \{y^i, y^j f \mid i \geq 1, j \geq 0\}$ . In fact, it follows from (a)–(d) and Lemma 4.16 that

- (e) the elements  $y, y^2, y^3, \dots, y^k, f, yf, y^2 f, \dots, y^{k-1} f$  of  $U$  are distinct.

Now it is routinely checked that the set

$$J = \{y^n, y^m f \mid n \geq k, m \geq k-1\}$$

is an ideal of  $U$ . By (e), the set  $U \setminus J$  consists of the elements

$$y, y^2, y^3, \dots, y^{k-1}, f, yf, y^2 f, \dots, y^{k-2} f.$$

Therefore,  $O_k \cong U/J$  by Lemma 4.14, whence  $\langle\langle O_k \rangle\rangle \subseteq \langle\langle T \rangle\rangle \vee \mathbf{RZ}$ .  $\square$

**Theorem 4.18.** (i) *For each  $k \geq 2$ , the pseudovariety  $\langle\langle O_k \rangle\rangle^{\text{bar}}$  is ji and*

$$\text{Excl}(O_k^{\text{bar}}) = \llbracket (\mathbf{ec}(a^\omega b)^{k-1})^\omega \approx (\mathbf{ec}((a^\omega b)^{k-1})^{\omega+1})^\omega \rrbracket, \quad (4.5)$$

*where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{a, b, c\}^+}$ .*

(ii) *The pseudovarieties  $\langle\langle O_2 \rangle\rangle^{\text{bar}}, \langle\langle O_3 \rangle\rangle^{\text{bar}}, \langle\langle O_4 \rangle\rangle^{\text{bar}}, \dots$  are all distinct.*

*Proof.* (i) Let  $\varphi$  denote the substitution into  $O_k^{\text{bar}}$  given by  $a \mapsto e$ ,  $b \mapsto x$ , and  $c \mapsto \bar{1}$ . Then  $(\mathbf{ec}(a^\omega b)^{k-1})^\omega \varphi = \overline{x^{k-1}}$  and  $(\mathbf{ec}((a^\omega b)^{k-1})^{\omega+1})^\omega \varphi = \overline{x^k}$ , and these are different elements of  $O_k^{\text{bar}}$ . Therefore the semigroup  $O_k^{\text{bar}}$  violates the pseudoidentity in (4.5).

It remains to assume that a semigroup  $T$  violates the pseudoidentity in (4.5), and then show that  $O_k^{\text{bar}} \in \langle\langle T \rangle\rangle$ . Replacing  $T$  by a subsemigroup if necessary, generality is not lost by assuming the existence of a surjective homomorphism  $\psi: \widehat{\{a, b, c\}^+} \rightarrow T$  such that

$$(\mathbf{ec}(a^\omega b)^{k-1})^\omega \psi \neq (\mathbf{ec}((a^\omega b)^{k-1})^{\omega+1})^\omega \psi.$$

Put  $f = a^\omega \psi$  and  $y = (a^\omega b) \psi$  and note that  $f^2 = f$  and  $fy = y$ .

The semigroup  $T$  acts on the right of the set  $B$  of  $\mathcal{L}$ -classes of its minimal ideal  $J$ ; denote the corresponding faithful transformation semigroup by  $(B, \text{RLM}(T))$ . Note that  $(B, \text{RLM}(T)) = \overline{(B, \text{RLM}(T))}$  because if  $b \in B$ , then any element of  $T$  in the  $\mathcal{L}$ -class of  $b$  acts on  $B$  as a constant map to  $b$  by the structure of completely simple semigroups. Consequently,  $\langle\langle \text{RLM}(T) \rangle\rangle^{\text{bar}} = \langle\langle \text{RLM}(T) \rangle\rangle$  by Corollary 2.6 since the constant mappings form the minimal ideal of  $\text{RLM}(T)$ .

Since  $(\mathbf{ec})\psi$  is in the minimal ideal  $J$  of  $T$ , it follows that the elements  $((\mathbf{ec})\psi)y^{k-1}$  and  $((\mathbf{ec})\psi)(y^{k-1})^{\omega+1}$  are  $\mathcal{R}$ -equivalent. However, they are not  $\mathcal{L}$ -equivalent because otherwise they would be  $\mathcal{H}$ -equivalent and hence

have the same idempotent power, as  $J$  is completely simple. Consequently,  $\text{RLM}(T)$  is nontrivial and so it follows from Proposition 2.5 that  $\langle\langle \text{RLM}(T) \rangle\rangle = \langle\langle \text{RLM}(T) \rangle\rangle \vee \mathbf{RZ}$ . Also, if  $z$  denotes the image of  $y$  under the quotient map  $T \rightarrow \text{RLM}(T)$  and  $d$  denotes the image of  $f$  under this map, then  $d^2 = d$ ,  $dz = z$ , and  $z^{k-1}$  is not a group element (as  $z^{k-1}$  and  $(z^{k-1})^{\omega+1}$  act differently on the  $\mathcal{L}$ -class of  $(ec)\psi$ ). Thus Lemma 4.17 implies that  $O_k \in \langle\langle \text{RLM}(T) \rangle\rangle \vee \mathbf{RZ} = \langle\langle \text{RLM}(T) \rangle\rangle$ . Consequently,  $O_k^{\text{bar}} \in \langle\langle \text{RLM}(T) \rangle\rangle^{\text{bar}} = \langle\langle \text{RLM}(T) \rangle\rangle \subseteq \langle\langle T \rangle\rangle$ .

(ii) This holds because for each  $k \geq 2$ , the semigroup  $O_k^{\text{bar}}$  satisfies the identity  $x^{k+1} \approx x^k$  but violates the identity  $x^k \approx x^{k-1}$ .  $\square$

**4.6. A sufficient condition for the join irreducibility of groups.** Recall that a normal subgroup  $N$  of a group  $G$  *splits* if there exists a subgroup  $K$  of  $G$  so that  $N \cap K = \{1\}$  and  $NK = G$ .

**Theorem 4.19** (G.M. Bergman, private communication, 2014). *Suppose that  $G$  is any finite sdi group with an abelian monolith  $N$  that splits. Then  $G$  is ji.*

*Proof.* By assumption, there exists a subgroup  $K$  of  $G$  with  $N \cap K = \{1\}$  and  $NK = G$ . Seeking a contradiction, suppose there exist finite groups  $G_1$  and  $G_2$  and some surjective homomorphism  $f$  from a subgroup  $H$  of  $G_1 \times G_2$  onto  $G$  such that  $G \notin \langle\langle G_1 \rangle\rangle$  and  $G \notin \langle\langle G_2 \rangle\rangle$ . Clearly we can assume that  $H\pi_j = G_j$  for the projection maps  $\pi_j : G_1 \times G_2 \rightarrow G_j$ , that is,  $H$  is a subdirect product of  $G_1$  and  $G_2$ . Further, we may assume that  $G_1$ ,  $G_2$ , and  $H$  are chosen so that the order of  $H$  is minimal.

Let  $H_2 = \{h_2 \in G_2 \mid (1, h_2) \in H\} \cong \ker(\pi_1) \cap H$ . If  $H_2$  is trivial, then  $\pi_1$  is injective on  $H$ , so that  $H \cong G_1$ , whence the contradiction  $G \prec G_1$  is obtained. Hence  $H_2$  is nontrivial. Observe that

(†) if  $L$  is a subgroup of  $H_2$  such that  $\{1\} \times L \trianglelefteq H$ , then  $L \trianglelefteq G_2$ ;

in particular,  $H_2 \trianglelefteq G_2$ . Indeed, if  $\ell \in L$  and  $g_2 \in G_2$ , then choosing any  $g_1 \in G_1$  with  $(g_1, g_2) \in H$ , we have

$$(1, g_2 \ell g_2^{-1}) = (g_1, g_2)(1, \ell)(g_1, g_2)^{-1} \in \{1\} \times L$$

by normality of  $\{1\} \times L$  in  $H$ , whence  $g_2 \ell g_2^{-1} \in L$ .

Suppose that  $\ker(f)$  has nontrivial intersection with the subgroup  $\{1\} \times H_2$  of  $H$ , say  $\ker(f) \cap (\{1\} \times H_2) = \{1\} \times L$  for some  $L \subseteq G_2$ . Then  $L$  is normal in  $H_2$  and so also normal in  $G_2$  by (†). By dividing  $G_2$  by this intersection, we could contradictorily decrease the order of  $H$ . Therefore  $\ker(f)$  intersects  $\{1\} \times H_2$  trivially.

Similarly, defining  $H_1 = \{h_1 \in G_1 \mid (h_1, 1) \in H\}$ , we have  $\{1\} \neq H_1 \trianglelefteq G_1$  and  $\ker(f)$  intersects  $H_1 \times \{1\}$  trivially. Then  $H_1 \times \{1\}$ ,  $\{1\} \times H_2$ , and  $\ker(f)$  are all normal in  $H$  and have pairwise trivial intersections.

Note that the centralizer of  $N$  in  $G$  is  $N$ . Indeed, since  $N$  is the unique minimal normal subgroup of  $G$ , the action of  $K$  on  $N$  by conjugation is faithful (otherwise, the kernel would be a normal subgroup of  $G$  not containing  $N$ ). If  $kn$  centralizes  $N$  with  $k \in K$  and  $n \in N$ , then since  $N$  is abelian, we have that  $k$  centralizes  $N$  and hence  $k = 1$  by the previous observation.

From now on, identify  $H_1$  with  $H_1 \times \{1\}$  and  $H_2$  with  $\{1\} \times H_2$ . Then  $H_1$  and  $H_2$  are normal in  $H$  and commute elementwise. We claim now that

$H_1f = N = H_2f$ . Indeed, since  $f$  is injective on each of these subgroups and these subgroups are normal in  $H$ , we conclude that  $N$  is contained in  $H_1f \cap H_2f$ . Since  $H_1$  and  $H_2$  commute elementwise, both  $H_1f$  and  $H_2f$  are contained in the centralizer of  $N$ , which is  $N$ . We conclude that  $H_1f = N = H_2f$  and  $f$  restricts to an isomorphism of  $H_1$  and  $H_2$  with  $N$ .

Let  $H^* = Kf^{-1}$ . Then since  $N \cap K = \{1\}$ , it follows that  $H^* \cap H_1$  is a subgroup of  $\ker(f)$ . But  $\ker(f) \cap H_1$  is trivial, so that  $H^* \cap H_1 = \{1\}$ . Similarly,  $H^* \cap H_2 = \{1\}$ . Note that  $H^*H_1$  and  $H^*H_2$  are subgroups of  $H$  because  $H_1$  and  $H_2$  are normal. Also  $(H^*H_1)f = KN = G = (H^*H_2)f$  and so by minimality of  $H$ , we have  $H^*H_1 = H = H^*H_2$ . In particular,  $G_2 \cong H\pi_2 = (H^*H_1)\pi_2 = H^*\pi_2$  and so, since  $H^* \cap H_1 = \{1\}$ , we deduce that  $G_2 \cong H^*$ . Similarly,  $G_1 \cong H^*$ . Therefore  $G \prec G_1 \times G_1$  and so  $G \in \langle\langle G_1 \rangle\rangle$ , a contradiction.  $\square$

## 5. JOIN IRREDUCIBLE PSEUDOVARIETIES

The present section contains 15 subsections. Some background results are recorded in the first subsection, while the latter 14 subsections are devoted to the pseudovarieties generated by the following 14 semigroups:

$$\begin{aligned} \mathbb{Z}_{p^n}, \quad \mathbb{Z}_2^{\text{bar}}, \quad N_n, \quad N_n^I, \quad N_2^{\text{bar}}, \quad (N_2^{\text{bar}})^I, \\ L_2, \quad L_2^I, \quad L_2^{\text{bar}}, \quad A_0, \quad A_0^I, \quad A_2, \quad B_2, \quad \ell_3^{\text{bar}}. \end{aligned} \quad (5.1)$$

Each subsection that is concerned with a semigroup  $S$  from (5.1) begins with a theorem that establishes the *ji* property of  $\langle\langle S \rangle\rangle$  by exhibiting a pseudoidentity that defines the pseudovariety  $\text{Excl}(S)$ . A basis  $\Sigma_S$  of identities for the pseudovariety  $\langle\langle S \rangle\rangle$  and an identity  $\varepsilon_S$  that defines its maximal subpseudovariety  $\langle\langle S \rangle\rangle \cap \text{Excl}(S)$  are then given in a proposition. The pair  $(\Sigma_S, \varepsilon_S)$  can be used to easily test if a finite semigroup generates the *ji* pseudovariety  $\langle\langle S \rangle\rangle$ . Indeed, for any finite semigroup  $T$ ,

$$\begin{aligned} T \models \Sigma_S \text{ and } T \not\models \varepsilon_S &\iff \langle\langle T \rangle\rangle \subseteq \langle\langle S \rangle\rangle \text{ and } \langle\langle T \rangle\rangle \not\subseteq \langle\langle S \rangle\rangle \cap \text{Excl}(S) \\ &\iff \langle\langle T \rangle\rangle = \langle\langle S \rangle\rangle. \end{aligned}$$

The pairs  $(\Sigma_S, \varepsilon_S)$ , where  $S$  ranges over the semigroups from (5.1), will be used in Section 7 to locate all *ji* pseudovarieties generated by semigroups of order up to five.

**5.1. Preliminaries.** The free semigroup and free monoid over a countably infinite alphabet  $\mathcal{A}$  are denoted by  $\mathcal{A}^+$  and  $\mathcal{A}^*$ , respectively. Elements of  $\mathcal{A}$  are called *variables* while elements of  $\mathcal{A}^*$  are called *words*. For any word  $\mathbf{w} \in \mathcal{A}^+$ ,

- the number of times a variable  $x$  occurs in  $\mathbf{w}$  is denoted by  $\text{occ}(x, \mathbf{w})$ ;
- the *content* of  $\mathbf{w}$ , denoted by  $\text{con}(\mathbf{w})$ , is the set of variables occurring in  $\mathbf{w}$ , that is,  $\text{con}(\mathbf{w}) = \{x \in \mathcal{A} \mid \text{occ}(x, \mathbf{w}) \geq 1\}$ ;
- the *initial part* of  $\mathbf{w}$ , denoted by  $\text{ini}(\mathbf{w})$ , is the word obtained by retaining the first occurrence of each variable in  $\mathbf{w}$ ;
- the *final part* of  $\mathbf{w}$ , denoted by  $\text{fin}(\mathbf{w})$ , is the word obtained by retaining the last occurrence of each variable in  $\mathbf{w}$ .

**Lemma 5.1.** *Let  $\mathbf{u} \approx \mathbf{v}$  be any semigroup identity. Then*

- (i)  $\mathbb{Z}_n \models \mathbf{u} \approx \mathbf{v}$  if and only if  $\text{occ}(x, \mathbf{u}) \equiv \text{occ}(x, \mathbf{v}) \pmod{n}$  for all  $x \in \mathcal{A}$ ;



- (ii)  $N_n^I \models \mathbf{u} \approx \mathbf{v}$  if and only if for all  $x \in \mathcal{A}$ , either  $\text{occ}(x, \mathbf{u}) = \text{occ}(x, \mathbf{v})$  or  $\text{occ}(x, \mathbf{u}), \text{occ}(x, \mathbf{v}) \geq n$ ;
- (iii)  $L_2^I \models \mathbf{u} \approx \mathbf{v}$  if and only if  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$ ;
- (iv)  $R_2^I \models \mathbf{u} \approx \mathbf{v}$  if and only if  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v})$ .

*Proof.* These results are well known and easily established. For instance, parts (i) and (ii) follow from Almeida [1, Lemma 6.1.4] while parts (iii) and (iv) can be found in Petrich and Reilly [17, Theorem V.1.9, parts (viii) and (ix)].  $\square$

The *local* of a pseudovariety  $\mathbf{V}$ , denoted by  $\mathbb{L}\mathbf{V}$ , is the pseudovariety of all finite semigroups  $S$  such that  $eSe \in \mathbf{V}$  for all idempotents  $e \in S$ .

**Lemma 5.2** (Almeida [1, Exercise 10.10.1]). *Let  $S$  be any finite semigroup that is not a monoid. If the pseudovariety  $\langle\langle S \rangle\rangle$  is  $\text{ji}$ , then the pseudovariety  $\langle\langle S^I \rangle\rangle$  is also  $\text{ji}$  and  $\text{Excl}(S^I) = \mathbb{L}\text{Excl}(S)$ .*

**5.2. The pseudovariety  $\langle\langle \mathbb{Z}_n \rangle\rangle$ .** For any set  $\pi = \{p_1, p_2, p_3, \dots\}$  of primes, let  $\pi'$  denote the set of primes complementary to  $\pi$ . If  $p$  is a prime, then simply write  $p'$  instead of  $\{p\}'$ . For example,  $2'$  denotes the set of odd primes. Retaining the above notation, recall that in  $\widehat{\{x\}^+}$ , the sequence  $x^{(p_1 p_2 \dots p_n)^{n!}}$  converges to an element (independent of the enumeration of  $\pi$ ), denoted by  $x^{\pi^\omega}$ , with the following property: if  $s$  is an element of a finite semigroup  $S$ , then  $s^{\pi^\omega}$  is a generator of the  $\pi'$ -primary component of the finite cyclic group generated by  $s^{\omega+1}$ . Here we recall that for a finite abelian group  $A$ , the  $\pi'$ -primary component of  $A$  is the direct product of the  $p$ -Sylow subgroups of  $A$  with  $p \notin \pi$ . In this case,  $s^{(\pi')^\omega}$  will then be a generator of the  $\pi$ -primary component of  $\langle s^{\omega+1} \rangle$ ; see [20, Proposition 7.1.16].

**Theorem 5.3.** *For any prime  $p$  with  $n \geq 1$ , the pseudovariety  $\langle\langle \mathbb{Z}_{p^n} \rangle\rangle$  is  $\text{ji}$  and*

$$\text{Excl}(\mathbb{Z}_{p^n}) = \llbracket (x^{(p')^\omega})^{p^{n-1}} \approx x^\omega \rrbracket. \quad (5.2)$$

*Proof.* The cyclic group  $\mathbb{Z}_{p^n} = \langle \mathbf{g} \mid \mathbf{g}^{p^n} = 1 \rangle$  violates the pseudoidentity in (5.2) because  $(\mathbf{g}^{(p')^\omega})^{p^{n-1}} = \mathbf{g}^{p^{n-1}} \neq 1 = \mathbf{g}^\omega$ . Therefore if  $\mathbb{Z}_{p^n}$  belongs to some pseudovariety  $\mathbf{V}$ , then  $\mathbf{V}$  violates the pseudoidentity in (5.2).

Conversely, suppose that the pseudoidentity in (5.2) is violated by  $\mathbf{V}$ , say it is violated by  $S \in \mathbf{V}$ . Generality is not lost by assuming that  $S$  is generated by an element  $s$  such that  $(s^{(p')^\omega})^{p^{n-1}} \neq s^\omega$ . Replacing  $s$  by  $s^{\omega+1}$ , we may assume that  $S$  is, in fact, a cyclic group generated by  $s$  such that  $(s^{(p')^\omega})^{p^{n-1}} \neq 1$ . But then the  $p$ -primary component of  $S$  is a cyclic group of order  $p^m$  with  $m \geq n$ . Therefore  $\mathbb{Z}_{p^n}$  divides  $S$ , whence  $\mathbb{Z}_{p^n} \in \mathbf{V}$ .  $\square$

**Proposition 5.4.** *Let  $n \geq 1$ .*

- (i) *The identities satisfied by the group  $\mathbb{Z}_n$  are axiomatized by*

$$xy \approx yx, \quad x^n y \approx y.$$

- (ii) *The maximal subpseudovarieties of  $\langle\langle \mathbb{Z}_n \rangle\rangle$  are precisely  $\langle\langle \mathbb{Z}_d \rangle\rangle$ , where  $d$  ranges over all maximal proper divisors of  $n$ . Consequently, for any prime  $p$  with  $k \geq 1$ , the subpseudovariety of  $\langle\langle \mathbb{Z}_{p^k} \rangle\rangle$  defined by*

$$x^{p^{k-1}+1} \approx x$$

is the unique maximal subpseudovariety of  $\langle\langle \mathbb{Z}_{p^k} \rangle\rangle$ .

*Proof.* These results are well known and easily established. For instance, part (i) follows from Almeida [1, Corollary 6.1.5] while part (ii) follows from Petrich and Reilly [17, Lemma VIII.6.14].  $\square$

### 5.3. The pseudovariety $\langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$ .

**Theorem 5.5.** *The pseudovariety  $\langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$  is ji and*

$$\text{Excl}(\mathbb{Z}_2^{\text{bar}}) = \llbracket (\mathbf{e}yx^{(2')^\omega})^\omega \approx (\mathbf{e}yx^\omega)^\omega \rrbracket,$$

where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{x, y\}^+}$ .

*Proof.* This follows from Theorems 4.3 and 5.3.  $\square$

Alternately, Rhodes and Steinberg [20, Example 7.3.20] have shown that

$$\text{Excl}(\mathbb{Z}_2^{\text{bar}}) = \llbracket ((x^\omega \mathbf{e}x^\omega)^\omega x^{(2')^\omega})^\omega \approx (x^\omega \mathbf{e}x^\omega)^\omega \rrbracket,$$

where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{x, y\}^+}$ .

### Proposition 5.6.

(i) *The identities satisfied by the semigroup  $\mathbb{Z}_2^{\text{bar}}$  are axiomatized by*

$$x^3 \approx x, \quad xyxy \approx yx^2y.$$

(ii) *The subpseudovariety of  $\langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$  defined by the identity*

$$xyx \approx yx^2$$

*is the unique maximal subpseudovariety of  $\langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$ .*

*Proof.* This follows from the dual of Tishchenko [23, Proposition 3.16], where the variety generated by  $(\mathbb{Z}_2^{\text{bar}})^{\text{op}}$  is denoted there by  $\mathbf{W}_2$ .  $\square$

### 5.4. The pseudovariety $\langle\langle N_n \rangle\rangle$ .

**Theorem 5.7.** *For each  $n \geq 2$ , the pseudovariety  $\langle\langle N_n \rangle\rangle$  is ji and*

$$\text{Excl}(N_n) = \llbracket x^{\omega+n-1} \approx x^{n-1} \rrbracket. \quad (5.3)$$

*Proof.* The semigroup  $N_n = \langle \mathbf{a} \mid \mathbf{a}^n = 0 \rangle$  violates the pseudoidentity in (5.3) because  $\mathbf{a}^{\omega+n-1} = 0 \neq \mathbf{a}^{n-1}$ . Therefore if  $N_n$  belongs to some pseudovariety  $\mathbf{V}$ , then  $\mathbf{V}$  violates the pseudoidentity in (5.3).

Conversely, suppose that the pseudoidentity in (5.3) is violated by  $\mathbf{V}$ , say it is violated by  $S \in \mathbf{V}$ . Then there exists some  $a \in S$  such that  $a^{\omega+n-1} \neq a^{n-1}$ . If there exist some  $i \leq n-1$  and some  $j > i$  such that  $a^i = a^j$ , then  $a^{n-1} = a^{n-1-i}a^i = a^{n-1-i}a^j = a^{n-1}a^{j-i}$ , so that

$$a^{n-1} = a^{n-1}a^{j-i} = a^{n-1}a^{2(j-i)} = \dots = a^{n-1}a^{\omega(j-i)} = a^{n-1+\omega},$$

which is a contradiction. Hence the sets  $\{a\}, \{a^2\}, \dots, \{a^{n-1}\}, \{a^i \mid i \geq n\}$  are pairwise disjoint. It follows that  $J = \{a^i \mid i \geq n\}$  is an ideal of the monogenic subsemigroup  $\langle a \rangle$  of  $S$  such that  $\langle a \rangle / J \cong N_n$ . Consequently,  $N_n \in \langle\langle S \rangle\rangle \subseteq \mathbf{V}$ .  $\square$

**Proposition 5.8.** *Let  $n \geq 2$ .*

(i) The identities satisfied by the semigroup  $N_n$  are axiomatized by

$$xy \approx yx, \quad (5.4a)$$

$$x^n \approx y_1 y_2 \cdots y_n, \quad (5.4b)$$

$$x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \approx x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m} \quad (5.4c)$$

for all  $m \geq 1$  and  $e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m \geq 1$  such that

(a)  $e = f < n$  where  $e = \sum_{i=1}^m e_i$  and  $f = \sum_{i=1}^m f_i$ ;

(b) for each  $k \in \{1, 2, \dots, m\}$ , either  $e_k = f_k$  or  $e + e_k, f + f_k \geq n$ .

(ii) The subpseudovariety of  $\langle\langle N_n \rangle\rangle$  defined by the identity

$$x^n \approx x^{n-1}$$

is the unique maximal subpseudovariety of  $\langle\langle N_n \rangle\rangle$ .

*Proof.* (i) Let  $\mathbf{u} \approx \mathbf{v}$  be any identity satisfied by the semigroup  $N_n$ . Generality is not lost by assuming that  $|\mathbf{u}| \leq |\mathbf{v}|$ . There are four cases to consider.

CASE 1.  $n \leq |\mathbf{u}| \leq |\mathbf{v}|$ . Then  $\mathbf{u} \approx \mathbf{v}$  is clearly deducible from (5.4b).

CASE 2.  $|\mathbf{u}| < n \leq |\mathbf{v}|$ . Let  $\varphi : \mathcal{A} \rightarrow N_n$  denote the substitution that maps all variables to  $\mathbf{a}$ . Then  $\mathbf{u}\varphi = \mathbf{a}^{|\mathbf{u}|} \neq 0$  and  $\mathbf{v}\varphi = \mathbf{a}^{|\mathbf{v}|} = 0$  imply the contradiction  $\mathbf{u}\varphi \neq \mathbf{v}\varphi$ . So the present case is impossible.

CASE 3.  $|\mathbf{u}| < |\mathbf{v}| < n$ . Then the contradiction  $\mathbf{u}\varphi = \mathbf{a}^{|\mathbf{u}|} \neq \mathbf{a}^{|\mathbf{v}|} = \mathbf{v}\varphi$  is obtained. Hence the present case is impossible.

CASE 4.  $|\mathbf{u}| = |\mathbf{v}| < n$ . Suppose that  $\text{con}(\mathbf{u}) \neq \text{con}(\mathbf{v})$ . Then by symmetry, it suffices to assume the existence of some  $z \in \text{con}(\mathbf{u}) \setminus \text{con}(\mathbf{v})$ . By letting  $\chi : \mathcal{A} \rightarrow N_n$  denote the substitution that maps  $z$  to 0 and all other variables to  $\mathbf{a}$ , the contradiction  $\mathbf{u}\chi = 0 \neq \mathbf{a}^{|\mathbf{v}|} = \mathbf{v}\chi$  is deduced. Therefore  $\text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$ , say

$$\text{con}(\mathbf{u}) = \text{con}(\mathbf{v}) = \{x_1, x_2, \dots, x_m\} \subseteq \mathcal{A}.$$

The identity (5.4a) can then be applied to convert the words  $\mathbf{u}$  and  $\mathbf{v}$  into

$$\mathbf{u}' = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{v}' = x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m}$$

respectively, where  $e_i = \text{occ}(x_i, \mathbf{u}) \geq 1$  and  $f_i = \text{occ}(x_i, \mathbf{v}) \geq 1$  for all  $i$ . Since

$$e = \sum_{i=1}^m e_i = |\mathbf{u}'| = |\mathbf{u}| \quad \text{and} \quad f = \sum_{i=1}^m f_i = |\mathbf{v}'| = |\mathbf{v}|,$$

the assumption of the present case implies that  $e = f < n$ , whence the identity  $\mathbf{u}' \approx \mathbf{v}'$  satisfies (a). Suppose that  $e_k \neq f_k$  for some  $k$  so that  $e + e_k \neq f + f_k$ . Let  $\psi : \mathcal{A} \rightarrow N_n$  denote the substitution that maps  $x_k$  to  $\mathbf{a}^2$  and all other variables to  $\mathbf{a}$ . Then  $\mathbf{u}'\psi = \mathbf{v}'\psi$  in  $N_n$ , where

$$\mathbf{u}'\psi = \left( \prod_{i=1}^{k-1} \mathbf{a}^{e_i} \right) (\mathbf{a}^2)^{e_k} \left( \prod_{i=k+1}^m \mathbf{a}^{e_i} \right) = \mathbf{a}^{e+e_k}$$

and similarly,  $\mathbf{v}'\psi = \mathbf{a}^{f+f_k}$ . Hence  $\mathbf{a}^{e+e_k} = \mathbf{a}^{f+f_k}$ . But  $e+e_k \neq f+f_k$  implies that  $e+e_k, f+f_k \geq n$ . Therefore the identity  $\mathbf{u}' \approx \mathbf{v}'$  also satisfies (b) and is deducible from (5.4c). The identity  $\mathbf{u} \approx \mathbf{v}$  is thus deducible from  $\{(5.4a), (5.4c)\}$ .

Consequently, the identity  $\mathbf{u} \approx \mathbf{v}$  is deducible from the identities (5.4). Conversely, it is routinely verified that the semigroup  $N_n$  satisfies (5.4).

(ii) This follows from Theorem 5.7 and part (i).  $\square$

### 5.5. The pseudovariety $\langle\langle N_n^I \rangle\rangle$ .

**Theorem 5.9.** *For any  $n \geq 1$ , the pseudovariety  $\langle\langle N_n^I \rangle\rangle$  is ji and*

$$\text{Excl}(N_n^I) = \mathbb{L} \text{Excl}(N_n) = \llbracket h^\omega(xh^\omega)^{\omega+n-1} \approx h^\omega(xh^\omega)^{n-1} \rrbracket.$$

*Proof.* For  $n = 1$ , the result follows from [20, Table 7.2] because  $N_1^I \cong Sl_2$ . For  $n \geq 2$ , the result follows from Lemma 5.2 and Theorem 5.7.  $\square$

**Proposition 5.10.** *Let  $n \geq 1$ .*

- (i) *The identities satisfied by the semigroup  $N_n^I$  are axiomatized by*

$$x^{n+1} \approx x^n, \quad xy \approx yx.$$

- (ii) *The subpseudovariety of  $\langle\langle N_n^I \rangle\rangle$  defined by the identity*

$$x^n y^{n-1} \approx x^{n-1} y^n$$

*is the unique maximal subpseudovariety of  $\langle\langle N_n^I \rangle\rangle$ .*

*Proof.* (i) This easily established result is well known; see, for example, Almeida [1, Corollary 6.1.5].

- (ii) This follows from Theorem 5.9 and part (i).  $\square$

### 5.6. The pseudovariety $\langle\langle N_2^{\text{bar}} \rangle\rangle$ .

**Theorem 5.11.** *The pseudovariety  $\langle\langle N_2^{\text{bar}} \rangle\rangle$  is ji and*

$$\text{Excl}(N_2^{\text{bar}}) = \llbracket (\mathbf{e}zx^{\omega+1})^\omega \approx (\mathbf{e}zx)^\omega \rrbracket,$$

*where  $\mathbf{e}$  is an idempotent from the minimal ideal of  $\widehat{\{x, z\}^+}$ .*

*Proof.* This follows from Theorems 4.3 and 5.7.  $\square$

**Proposition 5.12.**

- (i) *The identities satisfied by the semigroup  $N_2^{\text{bar}}$  are axiomatized by*

$$xyz \approx yz.$$

- (ii) *The subpseudovariety of  $\langle\langle N_2^{\text{bar}} \rangle\rangle$  defined by the identity*

$$xy \approx y^2$$

*is the unique maximal subpseudovariety of  $\langle\langle N_2^{\text{bar}} \rangle\rangle$ .*

*Proof.* (i) This follows from Tishchenko [23, Corollary 2.5(c) and Proposition 4.4].

- (ii) This follows from Tishchenko [23, Proposition 3.4].  $\square$

### 5.7. The pseudovariety $\langle\langle (N_2^{\text{bar}})^I \rangle\rangle$ .

**Theorem 5.13.** *The pseudovariety  $\langle\langle (N_2^{\text{bar}})^I \rangle\rangle$  is ji and*

$$\text{Excl}((N_2^{\text{bar}})^I) = \mathbb{L} \text{Excl}(N_2^{\text{bar}}) = \mathbb{L} \llbracket (\mathbf{e}zx^{\omega+1})^\omega \approx (\mathbf{e}zx)^\omega \rrbracket,$$

*where  $\mathbf{e}$  is an idempotent from the minimal ideal of  $\widehat{\{x, z\}^+}$ .*

*Proof.* This follows from Lemma 5.2 and Theorem 5.11.  $\square$

**Proposition 5.14.**

(i) The identities satisfied by the semigroup  $(N_2^{\text{bar}})^I$  are axiomatized by

$$\begin{aligned} x^3 &\approx x^2, & x^2hx &\approx xhx, & xhx^2 &\approx hx^2, & xhxtx &\approx hxtx, \\ xyxy &\approx yx^2y, & xyhxy &\approx yxhxy, & xyxty &\approx yx^2ty, & xyhxtx &\approx yxhxtx. \end{aligned}$$

(ii) The subpseudovariety of  $\langle\langle (N_2^{\text{bar}})^I \rangle\rangle$  defined by the identity

$$xyxyh^2 \approx x^2y^2h^2$$

is the unique maximal subpseudovariety of  $\langle\langle (N_2^{\text{bar}})^I \rangle\rangle$ .

*Proof.* This follows from the dual of Lee and Li [10, Corollary 6.6 and Lemma 6.7].  $\square$

### 5.8. The pseudovariety $\langle\langle L_2 \rangle\rangle$ .

**Theorem 5.15.** The pseudovariety  $\langle\langle L_2 \rangle\rangle$  is *ji* and

$$\text{Excl}(L_2) = \llbracket x^\omega (yx^\omega)^\omega \approx (yx^\omega)^\omega \rrbracket.$$

*Proof.* This result is dual to [1, Proposition 10.10.2(b)].  $\square$

**Proposition 5.16** (Rhodes and Steinberg [20, Table 7.1]).

(i) The identities satisfied by the semigroup  $L_2$  are axiomatized by

$$xy \approx x.$$

(ii) The pseudovariety  $\langle\langle L_2 \rangle\rangle$  is an atom in the lattice of pseudovarieties of semigroups.

### 5.9. The pseudovariety $\langle\langle L_2^I \rangle\rangle$ .

**Theorem 5.17.** The pseudovariety  $\langle\langle L_2^I \rangle\rangle$  is *ji* and

$$\text{Excl}(L_2^I) = \mathbb{L} \text{Excl}(L_2) = \llbracket h^\omega (xh^\omega)^\omega (yh^\omega (xh^\omega)^\omega)^\omega \approx h^\omega (yh^\omega (xh^\omega)^\omega)^\omega \rrbracket.$$

*Proof.* This follows from Lemma 5.2 and Theorem 5.15.  $\square$

**Proposition 5.18.**

(i) The identities satisfied by the semigroup  $L_2^I$  are axiomatized by

$$x^2 \approx x, \quad xyx \approx xy.$$

(ii) The subpseudovariety of  $\langle\langle L_2^I \rangle\rangle$  defined by the identity

$$xyz \approx xzy$$

is the unique maximal subpseudovariety of  $\langle\langle L_2^I \rangle\rangle$ .

*Proof.* This can be found in Almeida [1, Figure 5.1], where the pseudovariety  $\langle\langle L_2^I \rangle\rangle$  is denoted there by  $\mathbf{MK}_1$ .  $\square$

### 5.10. The pseudovariety $\langle\langle L_2^{\text{bar}} \rangle\rangle$ .

**Theorem 5.19.** *The pseudovariety  $\langle\langle L_2^{\text{bar}} \rangle\rangle$  is ji and*

$$\text{Excl}(L_2^{\text{bar}}) = \llbracket (\mathbf{e}zx^\omega(yx^\omega)^\omega)^\omega \approx (\mathbf{e}z(yx^\omega)^\omega)^\omega \rrbracket,$$

where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{x, y, z\}^+}$ .

*Proof.* This follows from Theorems 4.3 and 5.15.  $\square$

Alternately, Rhodes and Steinberg [20, Example 7.3.16] have shown that

$$\text{Excl}(L_2^{\text{bar}}) = \llbracket ((\mathbf{e}z)^\omega x^\omega (yx^\omega)^\omega)^\omega \approx ((\mathbf{e}z)^\omega (yx^\omega)^\omega)^\omega \rrbracket,$$

where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{x, y, z\}^+}$ .

**Proposition 5.20.**

- (i) *The identities satisfied by the semigroup  $L_2^{\text{bar}}$  are axiomatized by*

$$x^2 \approx x, \quad xyz \approx xzxyz.$$

- (ii) *The subpseudovariety of  $\langle\langle L_2^{\text{bar}} \rangle\rangle$  defined by the identity*

$$xyz \approx xzyz$$

*is the unique maximal subpseudovariety of  $\langle\langle L_2^{\text{bar}} \rangle\rangle$ .*

*Proof.* This can be found in Almeida [1, Figure 5.1], where the pseudovariety  $\langle\langle L_2^{\text{bar}} \rangle\rangle$  is denoted there by  $\llbracket R_3^\rho = Q_3^\rho \rrbracket_{\mathbf{B}}$ .  $\square$

### 5.11. The pseudovariety $\langle\langle A_0 \rangle\rangle$ .

**Theorem 5.21** (Lee [9, Proposition 2.3]). *The pseudovariety  $\langle\langle A_0 \rangle\rangle$  is ji and*

$$\text{Excl}(A_0) = \llbracket (x^\omega y^\omega)^{\omega+1} \approx x^\omega y^\omega \rrbracket.$$

**Proposition 5.22** (Lee [4, Section 4], Lee and Volkov [13, Theorem 4.1]).

- (i) *The identities satisfied by the semigroup  $A_0$  are axiomatized by*

$$x^3 \approx x^2, \quad xyx \approx xyxy, \quad xyx \approx yxyx.$$

- (ii) *The subpseudovariety of  $\langle\langle A_0 \rangle\rangle$  defined by the identity*

$$x^2 y^2 \approx y^2 x^2$$

*is the unique maximal subpseudovariety of  $\langle\langle A_0 \rangle\rangle$ .*

### 5.12. The pseudovariety $\langle\langle A_0^I \rangle\rangle$ .

**Theorem 5.23.** *The pseudovariety  $\langle\langle A_0^I \rangle\rangle$  is ji and*

$$\text{Excl}(A_0^I) = \mathbb{L} \text{Excl}(A_0) = \llbracket h^\omega ((xh^\omega)^\omega (yh^\omega)^\omega)^{\omega+1} \approx h^\omega (xh^\omega)^\omega (yh^\omega)^\omega \rrbracket.$$

*Proof.* This follows from Lemma 5.2 and Theorem 5.21.  $\square$

**Proposition 5.24** (Lee [7, Propositions 1.1 and 1.5(ii)]).

- (i) *The identities satisfied by the semigroup  $A_0^I$  are axiomatized by*

$$x^3 \approx x^2, \quad x^2 y x^2 \approx xyx, \quad xyxy \approx yxyx, \quad xyxzx \approx xyzx.$$

- (ii) *The subpseudovariety of  $\langle\langle A_0^I \rangle\rangle$  defined by the identity*

$$hx^2 y^2 h \approx hy^2 x^2 h$$

*is the unique maximal subpseudovariety of  $\langle\langle A_0^I \rangle\rangle$ .*

### 5.13. The pseudovariety $\langle\langle A_2 \rangle\rangle$ .

**Theorem 5.25** (Lee [5], Rhodes and Steinberg [20, Example 7.3.6]). *The pseudovariety  $\langle\langle A_2 \rangle\rangle$  is ji and*

$$\text{Excl}(A_2) = \llbracket ((x^\omega y)^\omega (yx^\omega)^\omega)^\omega \approx (x^\omega yx^\omega)^\omega \rrbracket.$$

**Proposition 5.26** (Lee [4, Theorem 2.7], Trahtman [24]).

(i) *The identities satisfied by the semigroup  $A_2$  are axiomatized by*

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad yxzx \approx xzxyx.$$

(ii) *The subpseudovariety of  $\langle\langle A_2 \rangle\rangle$  defined by the identity*

$$x^2 y^2 x^2 \approx x^2 y x^2$$

*is the unique maximal subpseudovariety of  $\langle\langle A_2 \rangle\rangle$ .*

### 5.14. The pseudovariety $\langle\langle B_2 \rangle\rangle$ .

**Theorem 5.27** (Rhodes and Steinberg [20, Example 7.3.4]). *The pseudovariety  $\langle\langle B_2 \rangle\rangle$  is ji and*

$$\text{Excl}(B_2) = \llbracket ((xy)^\omega (yx)^\omega (xy)^\omega)^\omega \approx (xy)^\omega \rrbracket.$$

**Proposition 5.28** (Lee [4, Theorem 3.6], Lee and Volkov [13, Proposition 3.5]).

(i) *The identities satisfied by the semigroup  $B_2$  are axiomatized by*

$$x^3 \approx x^2, \quad xyxyx \approx xyx, \quad x^2 y^2 \approx y^2 x^2.$$

(ii) *The subpseudovariety of  $\langle\langle B_2 \rangle\rangle$  defined by the identity*

$$xy^2 x \approx xyx$$

*is the unique maximal subpseudovariety of  $\langle\langle B_2 \rangle\rangle$ .*

### 5.15. The pseudovariety $\langle\langle \ell_3^{\text{bar}} \rangle\rangle$ .

**Theorem 5.29.** *The pseudovariety  $\langle\langle \ell_3^{\text{bar}} \rangle\rangle$  is ji and*

$$\text{Excl}(\ell_3^{\text{bar}}) = \llbracket (\mathbf{e}zx^\omega y)^\omega \approx (\mathbf{e}z(x^\omega y)^{\omega+1})^\omega \rrbracket,$$

*where  $\mathbf{e}$  is an idempotent in the minimal ideal of  $\widehat{\{x, y, z\}^+}$ .*

*Proof.* This is a special case of Theorem 4.18 since  $\ell_3^{\text{bar}} \cong O_2^{\text{bar}}$ .  $\square$

The remainder of this subsection is devoted to establishing a basis for the identities satisfied by  $\ell_3^{\text{bar}}$ . It turns out that it is notationally simpler to consider the dual semigroup  $(\ell_3^{\text{bar}})^{\text{op}} = \{a, b, c, d, e\}$ , given by the following multiplication table:

$(\ell_3^{\text{bar}})^{\text{op}}$	$a$	$b$	$c$	$d$	$e$
$a$	$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$b$	$a$	$d$
$c$	$a$	$a$	$c$	$a$	$e$
$d$	$d$	$d$	$d$	$d$	$d$
$e$	$e$	$e$	$e$	$e$	$e$

**Proposition 5.30.**

(i) The identities satisfied by the semigroup  $(\ell_3^{\text{bar}})^{\text{op}}$  are axiomatized by

$$xy^2 \approx xy, \quad xyz \approx xyzy. \quad (5.5)$$

(ii) The subpseudovariety of  $\langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle$  defined by the identity

$$xyzx \approx xyxz \quad (5.6)$$

is the unique maximal subpseudovariety of  $\langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle$ .

**Remark 5.31.** It is routinely shown that the semigroup  $(\ell_3^{\text{bar}})^{\text{op}}$  satisfies the identities (5.5) but violates the identity (5.6).

In this subsection, a word  $\mathbf{w}$  is said to be in *canonical form* if either

(CF1)  $\mathbf{w} = x_0x_1 \cdots x_m$  or

(CF2)  $\mathbf{w} = x_0x_1 \cdots x_k \cdot x_0 \cdot x_{k+1}x_{k+2} \cdots x_m$ ,

where  $x_0, x_1, \dots, x_m$  are distinct variables with  $0 \leq k \leq m$ .

**Remark 5.32.** Note the extreme cases for the word  $\mathbf{w}$  in (CF2):

(i) if  $0 = k = m$ , then  $\mathbf{w} = x_0^2$ ;

(ii) if  $0 = k < m$ , then  $\mathbf{w} = x_0^2x_1 \cdots x_m$ ;

(iii) if  $0 < k = m$ , then  $\mathbf{w} = x_0x_1x_2 \cdots x_mx_0$ .

**Lemma 5.33.** Given any word  $\mathbf{w}$ , the identities (5.5) can be used to convert  $\mathbf{w}$  into some word  $\mathbf{w}'$  in canonical form with  $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ .

*Proof.* Suppose that  $\text{ini}(\mathbf{w}) = x_0x_1 \cdots x_m$ . Then  $\mathbf{w}$  can be written as

$$\mathbf{w} = \prod_{i=0}^m (x_i \mathbf{w}_i) = x_0 \mathbf{w}_0 x_1 \mathbf{w}_1 \cdots x_m \mathbf{w}_m,$$

where  $\mathbf{w}_i \in \{x_0, x_1, \dots, x_i\}^*$  for all  $i$ . The identities (5.5) can be used to eliminate all occurrences of  $x_1, x_2, \dots, x_m$  from each  $\mathbf{w}_i$ , resulting in the word

$$\mathbf{w}' = \prod_{i=0}^m (x_i x_0^{e_i}) = x_0 x_0^{e_0} x_1 x_0^{e_1} \cdots x_m x_0^{e_m},$$

where  $e_0, e_1, \dots, e_m \geq 0$ . If  $e_0 = e_1 = \cdots = e_m = 0$ , then the word  $\mathbf{w}'$  is in canonical form (CF1) such that  $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}')$ . If  $k \geq 0$  is the least index such that  $e_k \geq 1$ , then  $e_0 = e_1 = \cdots = e_{k-1} = 0$ , so that

$$\mathbf{w}' = \left( \prod_{i=0}^{k-1} x_i \right) x_k x_0^{e_k} \left( \prod_{i=k+1}^m (x_i x_0^{e_i}) \right) \stackrel{(5.5)}{\approx} \underbrace{\left( \prod_{i=0}^{k-1} x_i \right) x_k x_0 \left( \prod_{i=k+1}^m x_i \right)}_{\mathbf{w}''}.$$

The word  $\mathbf{w}''$  is in canonical form (CF2) with  $\text{ini}(\mathbf{w}) = \text{ini}(\mathbf{w}'')$ .  $\square$

*Proof of Proposition 5.30(ii).* As observed in Remark 5.31, the semigroup  $(\ell_3^{\text{bar}})^{\text{op}}$  violates the identity (5.6). Hence  $\langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle \cap \llbracket (5.6) \rrbracket$  is a proper subpseudovariety of  $\langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle$ . It remains to show that each proper subpseudovariety  $\mathbf{V}$  of  $\langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle$  satisfies the identity (5.6). Since  $\mathbf{V} \neq \langle\langle \ell_3^{\text{bar}} \rangle^{\text{op}} \rangle$ , there exists an identity  $\mathbf{u} \approx \mathbf{v}$  of  $\mathbf{V}$  that is violated by  $(\ell_3^{\text{bar}})^{\text{op}}$ . Further, since the identities (5.5) are satisfied by  $(\ell_3^{\text{bar}})^{\text{op}}$  and so also by  $\mathbf{V}$ , it follows from Lemma 5.33 that the words  $\mathbf{u}$  and  $\mathbf{v}$  can be chosen to be in canonical form. There are two cases.



CASE 1.  $\text{ini}(\mathbf{u}) \neq \text{ini}(\mathbf{v})$ . Then by Theorem 5.17, the pseudovariety  $\mathbf{V}$  satisfies the pseudoidentity that defines  $\text{Excl}(L_2^I)$ . Since

$$h^\omega(xh^\omega)^\omega(yh^\omega(xh^\omega)^\omega)^\omega \stackrel{(5.5)}{\approx} h^2xy \quad \text{and} \quad h^\omega(yh^\omega(xh^\omega)^\omega)^\omega \stackrel{(5.5)}{\approx} h^2yx,$$

the pseudovariety  $\mathbf{V}$  satisfies the identity  $\alpha : h^2xy \approx h^2yx$ . Since

$$xyxz \stackrel{(5.5)}{\approx} xy^2xz \stackrel{\alpha}{\approx} xy^2zx \stackrel{(5.5)}{\approx} xyzx,$$

the pseudovariety  $\mathbf{V}$  satisfies the identity (5.6).

CASE 2.  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$  and  $\mathbf{u} \neq \mathbf{v}$ . If the words  $\mathbf{u}$  and  $\mathbf{v}$  are both of the form (CF1), then they are contradictorily equal. Hence either  $\mathbf{u}$  or  $\mathbf{v}$  is of the form (CF2). By symmetry, there are two subcases.

2.1.  $\mathbf{u}$  and  $\mathbf{v}$  are both of the form (CF2). Then

$$\mathbf{u} = x_0x_1 \cdots x_j \cdot x_0 \cdot x_{j+1}x_{j+2} \cdots x_m$$

$$\text{and } \mathbf{v} = x_0x_1 \cdots x_k \cdot x_0 \cdot x_{k+1}x_{k+2} \cdots x_m,$$

where  $0 \leq j, k \leq m$ . Since  $j \neq k$ , it suffices to assume by symmetry that  $0 \leq j < k \leq m$ . Let  $\varphi$  denote the substitution given by  $x_0 \mapsto xy$ ,  $x_i \mapsto y$  for all  $i \in \{1, 2, \dots, j\}$ , and  $x_i \mapsto z$  otherwise. Then

$$\mathbf{u}\varphi = x_0\varphi \cdot (x_1 \cdots x_j)\varphi \cdot x_0\varphi \cdot (x_{j+1}x_{j+2} \cdots x_m)\varphi$$

$$= xy \cdot y^j \cdot xy \cdot z^{m-j} \stackrel{(5.5)}{\approx} xyxz \quad \text{and}$$

$$\mathbf{v}\varphi = x_0\varphi \cdot (x_1 \cdots x_j)\varphi \cdot (x_{j+1}x_{j+2} \cdots x_k)\varphi \cdot x_0\varphi \cdot (x_{k+1}x_{k+2} \cdots x_m)\varphi$$

$$= xy \cdot y^j \cdot z^{k-j} \cdot xy \cdot z^{m-k} \stackrel{(5.5)}{\approx} xyzx.$$

Therefore the identity (5.6) is deducible from (5.5) and  $\mathbf{u} \approx \mathbf{v}$ . The pseudovariety  $\mathbf{V}$  thus satisfies the identity (5.6).

2.2.  $\mathbf{u}$  is of the form (CF1) while  $\mathbf{v}$  is of the form (CF2). Then

$$\mathbf{u} = x_0x_1 \cdots x_m \quad \text{and} \quad \mathbf{v} = x_0x_1 \cdots x_j \cdot x_0 \cdot x_{j+1}x_{j+2} \cdots x_m.$$

Since

$$\mathbf{u}x_{m+1}x_0 = \overbrace{x_0x_1 \cdots x_mx_{m+1}x_0}^{\mathbf{u}'}$$

$$\text{and } \mathbf{v}x_{m+1}x_0 \stackrel{(5.5)}{\approx} \underbrace{x_0x_1 \cdots x_j \cdot x_0 \cdot x_{j+1}x_{j+2} \cdots x_mx_{m+1}}_{\mathbf{v}'},$$

the pseudovariety  $\mathbf{V}$  satisfies the identity  $\mathbf{u}' \approx \mathbf{v}'$ . Now  $\mathbf{u}'$  and  $\mathbf{v}'$  are distinct words in canonical form (CF2) such that  $\text{ini}(\mathbf{u}') = \text{ini}(\mathbf{v}')$ . Therefore the arguments in Subcase 2.1 can be repeated to show that  $\mathbf{V}$  satisfies the identity (5.6).  $\square$

*Proof of Proposition 5.30(i).* As observed in Remark 5.31, the identities (5.5) are satisfied by the semigroup  $(\ell_3^{\text{bar}})^{\text{op}}$ . Conversely, suppose that  $\mathbf{u} \approx \mathbf{v}$  is any identity satisfied by  $(\ell_3^{\text{bar}})^{\text{op}}$ . By Lemma 5.33, the identities (5.5) can be used to convert  $\mathbf{u}$  and  $\mathbf{v}$  into words  $\mathbf{u}'$  and  $\mathbf{v}'$  in canonical form. Since the subsemigroup  $\{a, c, e\}$  of  $(\ell_3^{\text{bar}})^{\text{op}}$  and the semigroup  $L_2^I$  are isomorphic, it follows from Lemma 5.1(iii) that  $\text{ini}(\mathbf{u}') = \text{ini}(\mathbf{v}')$ . Suppose that  $\mathbf{u}' \neq \mathbf{v}'$ . Then by repeating the arguments in Case 2 of the proof of Proposition 5.30(ii),

the identity (5.6) is deducible from (5.5) and  $\mathbf{u}' \approx \mathbf{v}'$ . Since the semi-group  $(\ell_3^{\text{bar}})^{\text{op}}$  satisfies the identities (5.5) and  $\mathbf{u}' \approx \mathbf{v}'$ , it also satisfies (5.6); but this is impossible by Remark 5.31. Therefore  $\mathbf{u}' = \mathbf{v}'$ . Since

$$\mathbf{u} \stackrel{(5.5)}{\approx} \mathbf{u}' = \mathbf{v}' \stackrel{(5.5)}{\approx} \mathbf{v},$$

the identity  $\mathbf{u} \approx \mathbf{v}$  is deducible from (5.5).  $\square$

## 6. NON-ji PSEUDOVARIETIES

The present section contains nine subsections, each of which establishes one or more sufficient conditions for finite semigroups to generate pseudovarieties that are not ji. Each of these sufficient conditions, given as a corollary of some general result, presents some finite set  $\Sigma$  of identities and some identities  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$  with the property that for any finite semigroup  $S$ ,

$$S \models \Sigma \text{ and } S \not\models \varepsilon_i \text{ for all } i \implies \langle\langle S \rangle\rangle \text{ is not ji.}$$

In most cases,  $\Sigma$  will be a basis of identities for some join  $\mathbf{V} = \bigvee_{i=1}^k \mathbf{V}_i$  of compact pseudovarieties  $\mathbf{V}_1, \mathbf{V}_2, \dots, \mathbf{V}_k$  that satisfy the pseudoidentities  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k$ , respectively.

Sufficient conditions developed in this section will be used in Section 7 to locate all non-ji pseudovarieties generated by semigroups of order up to five.

**6.1. The pseudovariety  $\langle\langle \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}}, N_3^I \rangle\rangle$ .** In this subsection, it is convenient to write

$$\mathcal{K} = \{\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}}, N_3^I\}.$$

**Proposition 6.1** (Lee and Li [11, Theorem 1.1 and Proposition 3.1]). *The identities satisfied by the semigroup  $\mathbb{Z}_3 \times \mathbb{Z}_4 \times \mathbb{Z}_2^{\text{bar}} \times (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \times N_3^I$  are axiomatized by*

$$\begin{aligned} x^{15} &\approx x^3, & x^{14}hx &\approx x^2hx, & x^{13}hx^2 &\approx xhx^2, & x^{13}hxtx &\approx xhxtx, \\ x^3hx &\approx xhx^3, & xhx^2tx &\approx x^3htx, \\ xhx^2y^2ty &\approx xhy^2x^2ty, \\ xhykxytxdy &\approx xhykxytxdy, & xhykxytydx &\approx xhykxytydx. \end{aligned} \tag{6.1}$$

**Corollary 6.2.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.1) but violates all of the identities*

$$x^3 \approx x, \quad xy \approx yx. \tag{6.2}$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle \mathcal{K} \rangle\rangle$  that is not ji.*

*Proof.* By Proposition 6.1, the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle \mathcal{K} \rangle\rangle = \langle\langle \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle \vee \langle\langle \mathbb{Z}_3, \mathbb{Z}_4, N_3^I \rangle\rangle$$

holds. But the two identities in (6.2) are satisfied by  $\mathbb{Z}_2^{\text{bar}} \times (\mathbb{Z}_2^{\text{bar}})^{\text{op}}$  and  $\mathbb{Z}_3 \times \mathbb{Z}_4 \times N_3^I$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_3, \mathbb{Z}_4, N_3^I \rangle\rangle$ .  $\square$

**Corollary 6.3.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.1) but violates all of the identities*

$$xy \approx yx, \quad xyx^2 \approx xy, \quad x^2yx \approx yx. \quad (6.3)$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle \mathcal{K} \rangle\rangle$  that is not  $\mathbf{ji}$ .*

*Proof.* By Proposition 6.1, the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle \mathcal{K} \rangle\rangle = \langle\langle \mathbb{Z}_3, \mathbb{Z}_4, N_3^I \rangle\rangle \vee \langle\langle (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle \vee \langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$$

holds. But the three identities in (6.3) are satisfied by  $\mathbb{Z}_3 \times \mathbb{Z}_4 \times N_3^I$ ,  $(\mathbb{Z}_2^{\text{bar}})^{\text{op}}$ , and  $\mathbb{Z}_2^{\text{bar}}$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_3, \mathbb{Z}_4, N_3^I \rangle\rangle$ ,  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle$ , and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$ .  $\square$

**Corollary 6.4.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.1) but violates all of the identities*

$$x^4 \approx x^3, \quad x^4y \approx y, \quad x^3y \approx y, \quad xyx^2 \approx xy, \quad x^2yx \approx yx. \quad (6.4)$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle \mathcal{K} \rangle\rangle$  that is not  $\mathbf{ji}$ .*

*Proof.* The inclusion  $\langle\langle S \rangle\rangle \subseteq \bigvee \{ \langle\langle T \rangle\rangle \mid T \in \mathcal{K} \}$  holds by Proposition 6.1. But the five identities in (6.4) are satisfied by  $N_3^I$ ,  $\mathbb{Z}_4$ ,  $\mathbb{Z}_3$ ,  $(\mathbb{Z}_2^{\text{bar}})^{\text{op}}$ , and  $\mathbb{Z}_2^{\text{bar}}$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle T \rangle\rangle$  for all  $T \in \mathcal{K}$ .  $\square$

**6.2. The pseudovariety  $\langle\langle \mathbb{Z}_m, N_n^I, L_2^I, R_2^I, A_0^I \rangle\rangle$ .** In this subsection, it is convenient to write

$$\mathcal{T}_{m,n} = \{ \mathbb{Z}_m, N_n^I, L_2^I, R_2^I, A_0^I \}.$$

and  $T_{m,n} = \mathbb{Z}_m \times N_n^I \times L_2^I \times R_2^I \times A_0^I$ .

**Proposition 6.5.** *Let  $m \geq 1$  and  $n \geq 2$ . Then the identities satisfied by the semigroup  $T_{m,n}$  are axiomatized by*

$$x^{m+n} \approx x^n, \quad x^{m+n-1}yx \approx x^{n-1}yx, \quad x^2yx \approx xyx^2, \quad xyxzx \approx x^2yzx. \quad (6.5)$$

**Remark 6.6.** (i) Since  $N_2$  is isomorphic to the subsemigroup  $\{0, \text{fe}\}$  of  $A_0$ , it follows that  $N_2^I \in \langle\langle A_0^I \rangle\rangle$ . Therefore  $\langle\langle T_{m,1} \rangle\rangle = \langle\langle T_{m,2} \rangle\rangle$ . This is the reason for the assumption  $n \geq 2$ .

(ii) The basic case  $(m, n) = (1, 2)$  for Proposition 6.5 was first established in Lee [8, Proposition 2.3(i)].

Suppose that a word  $\mathbf{w}$  can be written in the form

$$\mathbf{w} = \mathbf{w}_0 \prod_{i=1}^r (x^{e_i} \mathbf{w}_i) = \mathbf{w}_0 x^{e_1} \mathbf{w}_1 x^{e_2} \mathbf{w}_2 \cdots x^{e_r} \mathbf{w}_r,$$

where  $x \in \mathcal{A}$ ,  $\mathbf{w}_0, \mathbf{w}_r \in \mathcal{A}^*$ , and  $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_{r-1} \in \mathcal{A}^+$  are such that  $x \notin \text{con}(\mathbf{w}_0 \mathbf{w}_1 \cdots \mathbf{w}_r)$ , and  $e_1, e_2, \dots, e_r \in \{1, 2, \dots\}$ . Then the factors  $x^{e_1}, x^{e_2}, \dots, x^{e_r}$  are called  $x$ -stacks, or simply *stacks*, of  $\mathbf{w}$ . The *weight* of the  $x$ -stack  $x^{e_i}$  is  $e_i$ .

It is easily shown that the identities (6.5) can be used to convert any word into a word  $\mathbf{w}$  such that for each  $x \in \mathcal{A}$ ,

- (I) the number of  $x$ -stacks in  $\mathbf{w}$  is at most two;
- (II) if  $\mathbf{w}$  has one  $x$ -stack, then its weight is at most  $m + n - 1$ ;

- (III) if  $\mathbf{w}$  has two  $x$ -stacks, then the weight of the first  $x$ -stack is at most  $m + n - 2$  while the weight of the second  $x$ -stack is one.

In the present subsection, a word  $\mathbf{w}$  that satisfies (I)–(III) is said to be in *canonical form*. Note that if  $\mathbf{w}$  is a word in canonical form, then  $\text{occ}(x, \mathbf{w}) \leq m + n - 1$  for any  $x \in \mathcal{A}$ .

**Lemma 6.7.** *Let  $\mathbf{u}$  and  $\mathbf{v}$  be any words in canonical form such that the identity  $\mathbf{u} \approx \mathbf{v}$  is satisfied by the semigroup  $T_{m,n}$ . Then for any  $x \in \mathcal{A}$ ,*

- (i)  $\text{occ}(x, \mathbf{u}) \equiv \text{occ}(x, \mathbf{v}) \pmod{m}$ ;
- (ii) *either  $\text{occ}(x, \mathbf{u}) = \text{occ}(x, \mathbf{v}) \leq n$  or  $n < \text{occ}(x, \mathbf{u}), \text{occ}(x, \mathbf{v}) \leq m + n - 1$ .*

*Proof.* This follows from Lemma 5.1 parts (i) and (ii).  $\square$

For any word  $\mathbf{w}$  and any distinct variables  $x_1, x_2, \dots, x_r$ , let  $\mathbf{w}_{\{x_1, x_2, \dots, x_r\}}$  denote the word obtained from  $\mathbf{w}$  by retaining only the variables  $x_1, x_2, \dots, x_r$ . It is clear that any monoid that satisfies an identity  $\mathbf{u} \approx \mathbf{v}$  also satisfies  $\mathbf{u}_{\{x_1, x_2, \dots, x_r\}} \approx \mathbf{v}_{\{x_1, x_2, \dots, x_r\}}$  for any distinct variables  $x_1, x_2, \dots, x_r$ .

**Lemma 6.8.** *Let  $\mathbf{u}$  and  $\mathbf{v}$  be any words in canonical form such that the identity  $\mathbf{u} \approx \mathbf{v}$  is satisfied by the semigroup  $T_{m,n}$ . Then*

- (i) *for any distinct  $x, y \in \mathcal{A}$ , the identity  $\mathbf{u}_{\{x,y\}} \approx \mathbf{v}_{\{x,y\}}$  cannot be any of*

$$x^{e_1}y^{f_1} \approx x^{e_2}y^{f_2}x^{e_3}, \quad x^{e_1}y^{f_1} \approx y^{f_2}x^{e_2}y^{f_3}, \quad x^{e_1}y^{f_1} \approx x^{e_2}y^{f_2}x^{e_3}y^{f_3}, \quad (6.6)$$

*where  $e_1, f_1, e_2, f_2, e_3, f_3 \geq 1$ ;*

- (ii)  *$\mathbf{u}$  has two  $x$ -stacks if and only if  $\mathbf{v}$  has two  $x$ -stacks;*
- (iii)  *$x^e$  is the first  $x$ -stack of  $\mathbf{u}$  if and only if  $x^e$  is the first  $x$ -stack of  $\mathbf{v}$ .*

*Proof.* (i) The three identities in (6.6) are violated by the semigroups  $R_2^I$ ,  $L_2^I$ , and  $A_0^I$ , respectively.

- (ii) Suppose that  $\mathbf{u}$  has two  $x$ -stacks. Then by (III),

$$\mathbf{u} = \mathbf{u}_1 x^{e-1} \mathbf{u}_2 x \mathbf{u}_3$$

for some  $\mathbf{u}_1, \mathbf{u}_3 \in \mathcal{A}^*$  and  $\mathbf{u}_2 \in \mathcal{A}^+$  with  $x \notin \text{con}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3)$  and  $2 \leq e \leq m + n - 1$ . Seeking a contradiction, suppose that  $\mathbf{v}$  has only one  $x$ -stack. Then by (II) and part (i),

$$\mathbf{v} = \mathbf{v}_1 x^f \mathbf{v}_2$$

for some  $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}^*$  with  $x \notin \text{con}(\mathbf{v}_1 \mathbf{v}_2)$  and  $1 \leq f \leq m + n - 1$ . Since the word  $\mathbf{u}_2$  is nonempty, it contains some  $y$ -stack. Since  $\text{con}(\mathbf{u}) = \text{con}(\mathbf{v})$  by Lemma 5.1(ii), it follows that  $y \in \text{con}(\mathbf{v}_1 \mathbf{v}_2)$ . By symmetry, it suffices to assume that  $y \in \text{con}(\mathbf{v}_1)$ , so that  $\text{ini}(\mathbf{v}) = \dots y \dots x \dots$ . Since  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$  by Lemma 5.1(iii), it follows that  $y \in \text{con}(\mathbf{u}_1)$ . Hence the word  $\mathbf{u}$  contains two  $y$ -stacks, the first of which occurs in  $\mathbf{u}_1$  while the second occurs in  $\mathbf{u}_2$ . Therefore  $\text{fin}(\mathbf{u}) = \dots y \dots x \dots$ . Since  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v})$  by Lemma 5.1(iv), it follows that  $y \notin \text{con}(\mathbf{v}_2)$ . Consequently, the identity  $\mathbf{u}_{\{x,y\}} \approx \mathbf{v}_{\{x,y\}}$  is  $y^r x^{e-1} y x \approx y^s x^f$  for some  $r, s \geq 1$ , but this contradicts part (i).

- (iii) Let  $x^e$  be a first  $x$ -stack of  $\mathbf{u}$ . By part (ii), there are two cases.

CASE 1.  $\mathbf{u}$  and  $\mathbf{v}$  each has only one  $x$ -stack. Then by (II),

$$\mathbf{u} = \mathbf{u}_1 x^e \mathbf{u}_2 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_1 x^f \mathbf{v}_2$$

for some  $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2 \in \mathcal{A}^*$  with  $x \notin \text{con}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{v}_1 \mathbf{v}_2)$  and  $1 \leq e, f \leq m + n - 1$ . Since  $e = \text{occ}(x, \mathbf{u})$  and  $f = \text{occ}(x, \mathbf{v})$ , it follows from part (ii) that either  $e = f \leq n$  or  $n < e, f \leq m + n - 1$ . If  $n < e, f \leq m + n - 1$ , then  $e = f$  by part (i).

CASE 2.  $\mathbf{u}$  and  $\mathbf{v}$  each has two  $x$ -stacks. Then by (III),

$$\mathbf{u} = \mathbf{u}_1 x^{e-1} \mathbf{u}_2 x \mathbf{u}_3 \quad \text{and} \quad \mathbf{v} = \mathbf{v}_1 x^{f-1} \mathbf{v}_2 x \mathbf{v}_3$$

for some  $\mathbf{u}_1, \mathbf{u}_3, \mathbf{v}_1, \mathbf{v}_3 \in \mathcal{A}^*$  and  $\mathbf{u}_2, \mathbf{v}_2 \in \mathcal{A}^+$  with  $x \notin \text{con}(\mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3 \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3)$  and  $2 \leq e, f \leq m + n - 1$ . Since  $e = \text{occ}(x, \mathbf{u})$  and  $f = \text{occ}(x, \mathbf{v})$ , it follows from the same argument in Case 1 that  $e = f$ .  $\square$

**Lemma 6.9.** *Let  $\mathbf{u}$  and  $\mathbf{v}$  be any words in canonical form such that the identity  $\mathbf{u} \approx \mathbf{v}$  is satisfied by the semigroup  $T_{m,n}$ . Then the following are equivalent:*

- (a)  $\mathbf{u} \in \mathcal{A}^* x^e y^f \mathcal{A}^*$  where  $x^e$  and  $y^f$  are stacks of  $\mathbf{u}$ ;
- (b)  $\mathbf{v} \in \mathcal{A}^* x^e y^f \mathcal{A}^*$  where  $x^e$  and  $y^f$  are stacks of  $\mathbf{v}$ .

Further,  $x^e$  is the first  $x$ -stack of  $\mathbf{u}$  if and only if  $x^e$  is the first  $x$ -stack of  $\mathbf{v}$ , and  $y^f$  is the first  $y$ -stack of  $\mathbf{u}$  if and only if  $y^f$  is the first  $y$ -stack of  $\mathbf{v}$ .

*Proof.* First, note that  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$  and  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v})$  by Lemma 5.1. Suppose that (a) holds. Then

$$\mathbf{u} = \mathbf{u}_1 x^e y^f \mathbf{u}_2$$

for some  $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{A}^*$  such that  $\mathbf{u}_1$  does not end with  $x$  while  $\mathbf{u}_2$  does not begin with  $y$ . There are four cases depending on which of  $x^e$  and  $y^f$  are first stacks in  $\mathbf{u}$ .

CASE 1.  $x^e$  is the first  $x$ -stack in  $\mathbf{u}$  and  $y^f$  is the first  $y$ -stack in  $\mathbf{u}$ . Then  $x, y \notin \text{con}(\mathbf{u}_1)$ , so that  $\text{ini}(\mathbf{u}) = \cdots xy \cdots$ . By Lemma 6.8(iii),  $x^e$  is the first  $x$ -stack of  $\mathbf{v}$  and  $y^f$  is the first  $y$ -stack of  $\mathbf{v}$ . Since  $\text{ini}(\mathbf{v}) = \text{ini}(\mathbf{u}) = \cdots xy \cdots$ ,

$$\mathbf{v} = \mathbf{v}_1 x^e \mathbf{v}_2 y^f \mathbf{v}_3$$

for some  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathcal{A}^*$  such that  $x \notin \text{con}(\mathbf{v}_1)$  and  $y \notin \text{con}(\mathbf{v}_1 \mathbf{v}_2)$ , and that any stack of  $\mathbf{v}$  that occurs in  $\mathbf{v}_2$  cannot be a first stack. Suppose that  $\mathbf{v}_2 \neq \emptyset$ . Then the first variable  $z$  of  $\mathbf{v}_2$  constitutes the second  $z$ -stack of  $\mathbf{v}$ . Hence

$$\mathbf{v} = \underbrace{\cdots z^r \cdots}_{\mathbf{v}_1} x^e \underbrace{z \cdots}_{\mathbf{v}_2} y^f \mathbf{v}_3,$$

where  $z^r$  is the first  $z$ -stack of  $\mathbf{v}$ , and  $\text{ini}(\mathbf{v}) = \cdots z \cdots xy \cdots$ . By Lemma 6.8(ii), the word  $\mathbf{u}$  contains two  $z$ -stacks; by part (iii) of the same lemma, the first  $z$ -stack of  $\mathbf{u}$  is  $z^r$ . Since  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v}) = \cdots z \cdots xy \cdots$ , the  $z$ -stack  $z^r$  of  $\mathbf{u}$  occurs in  $\mathbf{u}_1$ :

$$\mathbf{u} = \underbrace{\cdots z^r \cdots}_{\mathbf{u}_1} x^e y^f \mathbf{u}_2.$$

The second  $z$ -stack of  $\mathbf{u}$  occurs in either  $\mathbf{u}_1$  or  $\mathbf{u}_2$ . There are two subcases.

- 1.1. The second  $z$ -stack of  $\mathbf{u}$  occurs in  $\mathbf{u}_1$ . Then  $\text{fin}(\mathbf{v}) = \text{fin}(\mathbf{u}) = \cdots z \cdots x \cdots$ , so that  $\mathbf{v}$  must contain a second  $x$ -stack occurring in either  $\mathbf{v}_2$  or  $\mathbf{v}_3$ . The identity  $\mathbf{u}_{\{x,z\}} \approx \mathbf{v}_{\{x,z\}}$  is thus  $z^{r+1}x^{e+1} \approx z^r x^e z x$ , which is impossible by Lemma 6.8(i).
- 1.2. The second  $z$ -stack of  $\mathbf{u}$  occurs in  $\mathbf{u}_2$ . Then  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v}) = \cdots z \cdots y \cdots$ , so that  $\mathbf{u}$  must contain a second  $y$ -stack occurring after the second  $z$ -stack:

$$\mathbf{u} = \underbrace{\cdots z^r \cdots}_{\mathbf{u}_1} x^e y^f \underbrace{\cdots z \cdots y \cdots}_{\mathbf{u}_2}.$$

The identity  $\mathbf{u}_{\{y,z\}} \approx \mathbf{v}_{\{y,z\}}$  is thus  $z^r y^f z y \approx z^{r+1} y^{f+1}$ , which is impossible by Lemma 6.8(i).

Since both subcases are impossible,  $\mathbf{v}_2 = \emptyset$ . Hence (b) holds.

CASE 2.  $x^e$  is the first  $x$ -stack in  $\mathbf{u}$  and  $y^f$  is the second  $y$ -stack in  $\mathbf{u}$ . Then  $f = 1$  by (III) and

$$\mathbf{u} = \underbrace{\cdots y^r \cdots}_{\mathbf{u}_1} x^e y \mathbf{u}_2,$$

where  $y^r$  is the first  $y$ -stack of  $\mathbf{u}$ . Since  $\text{ini}(\mathbf{v}) = \text{ini}(\mathbf{u}) = \cdots y \cdots x \cdots$ , it follows from Lemma 6.8 parts (i) and (iii) that

$$\mathbf{v} = \mathbf{v}_1 y^r \mathbf{v}_2 x^e \mathbf{v}_3 y \mathbf{v}_4$$

for some  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathcal{A}^*$ , where  $y^r$  is the first  $y$ -stack of  $\mathbf{v}$  and  $x^e$  is the first  $x$ -stack of  $\mathbf{v}$ . Suppose that  $\mathbf{v}_3 \neq \emptyset$ . Then  $\mathbf{v}_3$  contains some  $z$ -stack  $z^s$ :

$$\mathbf{v} = \mathbf{v}_1 y^r \mathbf{v}_2 x^e \underbrace{\cdots z^s \cdots}_{\mathbf{v}_3} y \mathbf{v}_4.$$

There are two subcases depending on whether  $z^s$  is the first or second  $z$ -stack in  $\mathbf{v}$ .

- 2.1.  $z^s$  is the first  $z$ -stack in  $\mathbf{v}$ . Then  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v}) = \cdots y \cdots x \cdots z \cdots$ , so that every  $z$  of  $\mathbf{u}$  occurs in  $\mathbf{u}_2$ . Hence  $\mathbf{u}_{\{y,z\}} \in y^{r+1}\{z\}^+$  and

$$\mathbf{v}_{\{y,z\}} = \begin{cases} y^r z^s y z & \text{if } \mathbf{v} \text{ has a second } z\text{-stack occurring in } \mathbf{v}_4, \\ y^r z^{s+1} y & \text{if } \mathbf{v} \text{ has a second } z\text{-stack occurring in } \mathbf{v}_3, \\ y^r z^s y & \text{if } \mathbf{v} \text{ has no second } z\text{-stack.} \end{cases}$$

But this is impossible by Lemma 6.8(i).

- 2.2.  $z^s$  is the second  $z$ -stack in  $\mathbf{v}$ . Then  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v}) = \cdots z \cdots y \cdots$ , so that every  $z$  of  $\mathbf{u}$  occurs in  $\mathbf{u}_1$ . Hence  $\mathbf{u}_{\{x,z\}} \in \{z\}^+ \{x\}^+$  and

$$\mathbf{v}_{\{x,z\}} \in \begin{cases} \{z\}^+ \{x\}^+ z^s \{x\}^* & \text{if the first } z\text{-stack of } \mathbf{v} \text{ occurs in } \mathbf{v}_1 \text{ or } \mathbf{v}_2, \\ \{x\}^+ \{z\}^+ \{x\}^* & \text{if the first } z\text{-stack of } \mathbf{v} \text{ occurs in } \mathbf{v}_3. \end{cases}$$

But this is impossible by Lemma 6.8(i).

Since both subcases are impossible,  $\mathbf{v}_3 = \emptyset$ . Hence (b) holds.

CASE 3.  $x^e$  is the second  $x$ -stack in  $\mathbf{u}$  and  $y^f$  is the first  $y$ -stack in  $\mathbf{u}$ . Then  $e = 1$  by (III) and

$$\mathbf{u} = \underbrace{\cdots x^r \cdots}_{\mathbf{u}_1} xy^f \mathbf{u}_2,$$

where  $x^r$  is the first  $x$ -stack of  $\mathbf{u}$  with and  $y \notin \text{con}(\mathbf{u}_1)$  and  $x \notin \text{con}(\mathbf{u}_2)$ . Since  $\text{ini}(\mathbf{v}) = \text{ini}(\mathbf{u}) = \cdots x \cdots y \cdots$ , it follows from Lemma 6.8 that

$$\mathbf{v} = \mathbf{v}_1 x^r \mathbf{v}_2 x \mathbf{v}_3 y^f \mathbf{v}_4$$

for some  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4 \in \mathcal{A}^*$  with  $x \notin \text{con}(\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \mathbf{v}_4)$  and  $y \notin \text{con}(\mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3)$ . Suppose that  $\mathbf{v}_3 \neq \emptyset$ . Then  $\mathbf{v}_3$  contains some  $z$ -stack  $z^s$ :

$$\mathbf{v} = \mathbf{v}_1 x^r \mathbf{v}_2 x \underbrace{\cdots z^s \cdots}_{\mathbf{v}_3} y^f \mathbf{v}_4.$$

There are two subcases depending on whether or not  $z^s$  is the first  $z$ -stack of  $\mathbf{v}$ .

3.1.  $z^s$  is the first  $z$ -stack of  $\mathbf{v}$ . Since  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v}) = \cdots z \cdots y \cdots$ , the first  $z$ -stack  $z^s$  of  $\mathbf{u}$  occurs in  $\mathbf{u}_1$ . On the other hand, since  $\text{fin}(\mathbf{u}) = \text{fin}(\mathbf{v}) = \cdots x \cdots z \cdots$ , the word  $\mathbf{u}$  must contain a second  $z$ -stack in  $\mathbf{u}_2$ , whence  $\mathbf{v}$  must also contain a second  $z$ -stack by Lemma 6.8(ii). Hence  $\mathbf{v}_{\{x,z\}} = x^{r+1} z^{s+1}$  and

$$\mathbf{u}_{\{x,z\}} = \begin{cases} z^s x^{r+1} z & \text{if } z^s \text{ in } \mathbf{u} \text{ occurs before the first } x\text{-stack,} \\ x^r z^s x z & \text{if } z^s \text{ in } \mathbf{u} \text{ occurs between the two } x\text{-stacks.} \end{cases}$$

But this is impossible by Lemma 6.8(i).

3.2.  $z^s$  is the second  $z$ -stack of  $\mathbf{v}$ . Then the identity  $\mathbf{u}_{\{y,z\}} \approx \mathbf{v}_{\{y,z\}}$  obtained by an argument symmetrical to the one in Subcase 3.1 produces a similar contradiction.

Since both subcases are impossible,  $\mathbf{v}_3 = \emptyset$ . Hence (b) holds.

CASE 4.  $x^e$  is the second  $x$ -stack in  $\mathbf{u}$  and  $y^f$  is the second  $y$ -stack in  $\mathbf{u}$ . Then (b) holds by an argument symmetrical to Case 1.

Therefore (b) holds in all four cases. By symmetry, (b) implies (a).  $\square$

*Proof of Proposition 6.5.* It is routinely verified that the semigroup  $T_{m,n}$  satisfies the identities (6.5). Conversely, suppose that  $\mathbf{u} \approx \mathbf{v}$  is any identity satisfied by the semigroup  $T_{m,n}$ . As observed earlier, the identities (6.5) can be used to convert  $\mathbf{u}$  and  $\mathbf{v}$  into words  $\mathbf{u}'$  and  $\mathbf{v}'$  in canonical form. By Lemma 6.7 parts (ii) and (iii), the words  $\mathbf{u}'$  and  $\mathbf{v}'$  share the same set of stacks. By Lemma 6.9, two stacks are adjacent in  $\mathbf{u}'$  if and only if they are adjacent in  $\mathbf{v}'$ . Therefore  $\mathbf{u}' = \mathbf{v}'$ . Since

$$\mathbf{u} \stackrel{(6.5)}{\approx} \mathbf{u}' = \mathbf{v}' \stackrel{(6.5)}{\approx} \mathbf{v},$$

the identity  $\mathbf{u} \approx \mathbf{v}$  is deducible from (6.5).  $\square$

**Corollary 6.10.** *Suppose that  $S$  is any finite semigroup that satisfies the identities*

$$x^{11} \approx x^5, \quad x^{10} y x \approx x^4 y x, \quad x^2 y x \approx x y x^2, \quad x y x z x \approx x^2 y z x \quad (6.7)$$

*but violates all of the identities*

$$x^2 \approx x, \quad x y x y \approx y x y x. \quad (6.8)$$

Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle \mathcal{T}_{6,5} \rangle\rangle$  that is not ji.

*Proof.* By Proposition 6.5 with  $m = 6$  and  $n = 5$ , the identities satisfied by  $T_{6,5}$  are axiomatized by (6.7). Hence the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle \mathcal{T}_{6,5} \rangle\rangle = \langle\langle L_2^I, R_2^I \rangle\rangle \vee \langle\langle \mathbb{Z}_6, N_5^I, A_0^I \rangle\rangle$$

holds. But the two identities in (6.8) are satisfied by  $L_2^I \times R_2^I$  and  $\mathbb{Z}_6 \times N_5^I \times A_0^I$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle L_2^I, R_2^I \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_6, N_5^I, A_0^I \rangle\rangle$ .  $\square$

**Corollary 6.11.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.7) but violates all of the identities*

$$x^6 \approx x^5, \quad x^6 y \approx y. \quad (6.9)$$

Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle \mathcal{T}_{6,5} \rangle\rangle$  that is not ji.

*Proof.* The argument in the proof of Corollary 6.10 implies the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle \mathcal{T}_{6,5} \rangle\rangle = \langle\langle N_5^I, L_2^I, R_2^I, A_0^I \rangle\rangle \vee \langle\langle \mathbb{Z}_6 \rangle\rangle.$$

The two identities in (6.9) are satisfied by  $N_5^I \times L_2^I \times R_2^I \times A_0^I$  and  $\mathbb{Z}_6$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_5^I, L_2^I, R_2^I, A_0^I \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle \mathbb{Z}_6 \rangle\rangle$ .  $\square$

### 6.3. Noncommutative nilpotent semigroups.

**Proposition 6.12.** *Any ji pseudovariety of nilpotent semigroups is commutative.*

*Proof.* Let  $\mathbf{V}$  be any ji pseudovariety of nilpotent semigroups. Then the inclusion  $\mathbf{V} \subseteq \mathbf{Com} \vee \mathbf{G}$  holds [1, Figure 9.1]. Since  $\mathbf{V}$  is ji and  $\mathbf{V} \not\subseteq \mathbf{G}$ , it follows that  $\mathbf{V} \subseteq \mathbf{Com}$ .  $\square$

**Corollary 6.13.** *Suppose that  $S$  is any finite semigroup that satisfies the identity*

$$x^6 \approx y_1 y_2 y_3 y_4 y_5 y_6$$

but violates the identity

$$xy \approx yx.$$

Then  $\langle\langle S \rangle\rangle$  is a pseudovariety of nilpotent semigroups that is not ji.

*Proof.* By assumption, the semigroup  $S$  is nilpotent and noncommutative. The result then holds by Proposition 6.12.  $\square$

### 6.4. The pseudovariety $\langle\langle N_{n+r}, N_n^I \rangle\rangle$ .

**Proposition 6.14.** *Let  $n, r \geq 1$ . Then the identities satisfied by the semigroup  $N_{n+r} \times N_n^I$  are axiomatized by*

$$xy \approx yx, \quad (6.10a)$$

$$x^{n+1} y_1 y_2 \cdots y_r \approx x^n y_1 y_2 \cdots y_r, \quad (6.10b)$$

$$x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \approx x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m} \quad (6.10c)$$

for all  $m \geq 1$  and  $e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m \geq 1$  such that

- (a)  $e = f < n + r$  where  $e = \sum_{i=1}^m e_i$  and  $f = \sum_{i=1}^m f_i$ ;
- (b) for each  $k \in \{1, 2, \dots, m\}$ , either
  - $e_k = f_k$  or
  - $e_k, f_k \geq n$  and  $e + e_k, f + f_k \geq n + r$ .



*Proof.* It is straightforwardly verified that the semigroup  $N_{n+r} \times N_n^I$  satisfies the identities (6.10). Conversely, let  $\mathbf{u} \approx \mathbf{v}$  be any identity satisfied by the semigroup  $N_{n+r} \times N_n^I$ . In view of Lemma 5.1(ii), the identity (6.10a) can be used to convert  $\mathbf{u}$  and  $\mathbf{v}$  into

$$\mathbf{u}' = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{v}' = x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m},$$

respectively, where  $e_i = \text{occ}(x_i, \mathbf{u})$  and  $f_i = \text{occ}(x_i, \mathbf{v})$  are such that either  $e_i = f_i$  or  $e_i, f_i \geq n$ . Let  $e = \sum_{i=1}^m e_i$  and  $f = \sum_{i=1}^m f_i$ . Generality is not lost by assuming that  $e \leq f$ . There are four cases to consider.

CASE 1.  $n+r \leq e \leq f$ . Choose any  $i \in \{1, 2, \dots, m\}$ . Suppose that  $e_i \neq f_i$ . Then as observed earlier,  $e_i, f_i \geq n$ . Hence

$$\begin{aligned} \mathbf{u}' &\stackrel{(6.10a)}{\approx} x_i^n \overbrace{x_1^{e_1} x_2^{e_2} \cdots x_{i-1}^{e_{i-1}} x_i^{e_i-n} x_{i+1}^{e_{i+1}} \cdots x_m^{e_m}}^{e-n \geq r \text{ variables}} \\ &\stackrel{(6.10b)}{\approx} x_i^{n+f_i} x_1^{e_1} x_2^{e_2} \cdots x_{i-1}^{e_{i-1}} x_i^{e_i-n} x_{i+1}^{e_{i+1}} \cdots x_m^{e_m} \\ &\stackrel{(6.10a)}{\approx} x_1^{e_1} x_2^{e_2} \cdots x_{i-1}^{e_{i-1}} x_i^{e_i+f_i} x_{i+1}^{e_{i+1}} \cdots x_m^{e_m}. \end{aligned}$$

Similarly,  $\mathbf{v}' \stackrel{(6.10)}{\approx} x_1^{f_1} x_2^{f_2} \cdots x_{i-1}^{f_{i-1}} x_i^{f_i+e_i} x_{i+1}^{f_{i+1}} \cdots x_m^{f_m}$ . Therefore the identities (6.10) can be used to convert  $\mathbf{u}'$  into  $\mathbf{v}'$ . It follows that  $\mathbf{u} \approx \mathbf{v}$  is deducible from (6.10).

CASE 2.  $e < n+r \leq f$ . Let  $\varphi : \mathcal{A} \rightarrow N_{n+r}$  be the substitution that maps all variables to  $\mathbf{a}$ . Then  $\mathbf{u}'\varphi = \mathbf{a}^e \neq 0$  and  $\mathbf{v}'\varphi = \mathbf{a}^f = 0$  imply the contradiction  $\mathbf{u}'\varphi \neq \mathbf{v}'\varphi$ . The present case is thus impossible.

CASE 3.  $e < f < n+r$ . Then the contradiction  $\mathbf{u}'\varphi = \mathbf{a}^e \neq \mathbf{a}^f = \mathbf{v}'\varphi$  is obtained. Hence the present case is impossible.

CASE 4.  $e = f < n+r$ . Suppose that  $e_k \neq f_k$  for some  $k$  so that  $e + e_k \neq f + f_k$ . Then as observed earlier,  $e_k, f_k \geq n$ . Let  $\psi : \mathcal{A} \rightarrow N_{n+r}$  be the substitution that maps  $x_k$  to  $\mathbf{a}^2$  and all other variables to  $\mathbf{a}$ . Then  $\mathbf{u}'\psi = \mathbf{v}'\psi$  in  $N_n$ , where

$$\mathbf{u}'\psi = \left( \prod_{i=1}^{k-1} \mathbf{a}^{e_i} \right) (\mathbf{a}^2)^{e_k} \left( \prod_{i=k+1}^m \mathbf{a}^{e_i} \right) = \mathbf{a}^{e+e_k}$$

and  $\mathbf{v}'\psi = \mathbf{a}^{f+f_k}$  similarly. Thus  $\mathbf{a}^{e+e_k} = \mathbf{a}^{f+f_k}$ . But  $e + e_k \neq f + f_k$  implies that  $e + e_k, f + f_k \geq n$ . Hence the identity  $\mathbf{u}' \approx \mathbf{v}'$  also satisfies (ii) and is deducible from (6.10c). The identity  $\mathbf{u} \approx \mathbf{v}$  is thus deducible from (6.10).  $\square$

**Corollary 6.15.** *Suppose that  $S$  is any finite semigroup that satisfies the identities*

$$xy \approx yx, \quad x^3 y_1 y_2 \approx x^2 y_1 y_2 \tag{6.11}$$

*but violates all of the identities*

$$x^3 \approx x^2, \quad x^2 y \approx xy^2. \tag{6.12}$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle N_4, N_2^I \rangle\rangle$  that is not ji.*

*Proof.* By Proposition 6.14 with  $n = r = 2$ , the identities satisfied by  $N_4 \times N_2^I$  are axiomatized by (6.11). Hence the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle N_4, N_2^I \rangle\rangle = \langle\langle N_4 \rangle\rangle \vee \langle\langle N_2^I \rangle\rangle$$

holds. But the two identities in (6.12) are satisfied by  $N_2^I$  and  $N_4$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_4 \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_2^I \rangle\rangle$ .  $\square$

**Corollary 6.16.** *Suppose that  $S$  is any finite semigroup that satisfies the identities*

$$xy \approx yx, \quad x^2yz \approx xy^2z, \quad x^2y_1y_2y_3y_4 \approx xy_1y_2y_3y_4 \quad (6.13)$$

*but violates all of the identities*

$$x^2 \approx x, \quad x^5 \approx y^5. \quad (6.14)$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle N_5, N_1^I \rangle\rangle$  that is not ji.*

*Proof.* By Proposition 6.14 with  $n = 1$  and  $r = 4$ , the identities satisfied by  $N_5 \times N_1^I$  are axiomatized by (6.13). Hence the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle N_5, N_1^I \rangle\rangle = \langle\langle N_5 \rangle\rangle \vee \langle\langle N_1^I \rangle\rangle$$

holds. But the two identities in (6.14) are satisfied by  $N_1^I$  and  $N_5$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_5 \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_1^I \rangle\rangle$ .  $\square$

**6.5. The pseudovariety  $\langle\langle N_n^I, N_2^{\text{bar}} \rangle\rangle$ .** It turns out to be notationally simpler to find a basis of identities for  $N_n^I \times (N_2^{\text{bar}})^{\text{op}}$  instead of  $N_n^I \times N_2^{\text{bar}}$ .

**Proposition 6.17.** *Let  $n \geq 2$ . Then the identities satisfied by the semigroup  $N_n^I \times (N_2^{\text{bar}})^{\text{op}}$  are axiomatized by*

$$x^{n+1} \approx x^n, \quad xyx^n \approx xyx^{n-1}, \quad (6.15a)$$

$$xyzt \approx xytz. \quad (6.15b)$$

In the present subsection, a word of length at least two is said to be in *canonical form* if it is either

(CF1)  $x^2 \cdot x^e z_1^{f_1} z_2^{f_2} \dots z_k^{f_k}$  or

(CF2)  $xy \cdot x^{e_1} y^{e_2} z_1^{f_1} z_2^{f_2} \dots z_k^{f_k}$ ,

where

- (I)  $x, y, z_1, z_2, \dots, z_k$  are distinct variables with  $k \geq 0$ ;
- (II)  $z_1, z_2, \dots, z_k$  are in alphabetical order;
- (III)  $e \in \{0, 1, \dots, n-2\}$ ,  $e_i \in \{0, 1, \dots, n-1\}$ , and  $f_i \in \{1, 2, \dots, n\}$ .

**Lemma 6.18.** *The identities (6.15) can be used to convert any word of length at least two into a word in canonical form.*

*Proof.* Let  $\mathbf{w}$  be any word of length at least two. Then  $\mathbf{w} = x^2\mathbf{u}$  or  $\mathbf{w} = xy\mathbf{u}$  for some distinct  $x, y \in \mathcal{A}$  and  $\mathbf{u} \in \mathcal{A}^*$ . The identity (6.15b) can first be used to rearrange the variables of the suffix  $\mathbf{u}$  until  $\mathbf{w}$  becomes a word of the form (CF1) or (CF2) with (I) and (II) satisfied. The identities (6.15a) can then be used to reduce the exponents  $e, e_i, f_i$  so that (III) is satisfied.  $\square$

**Lemma 6.19.** *The semigroup  $N_2^{\text{bar}}$  satisfies an identity  $\mathbf{u} \approx \mathbf{v}$  if and only if the words  $\mathbf{u}$  and  $\mathbf{v}$  share the same suffix of length two.*

*Proof.* This is routinely established and its dual result for  $(N_2^{\text{bar}})^{\text{op}}$  was observed by Lee and Li [10, Remark 6.2(i)].  $\square$

*Proof of Proposition 6.17.* It is easily verified, either directly or by Lemmas 5.1(ii) and 6.19, that the identities (6.15) are satisfied by the semigroup  $N_n^I \times (N_2^{\text{bar}})^{\text{op}}$ . Hence it remains to show that any identity  $\mathbf{u} \approx \mathbf{v}$  satisfied by the semigroup  $N_n^I \times (N_2^{\text{bar}})^{\text{op}}$  is deducible from the identities (6.15). It is easily shown that if either  $\mathbf{u}$  or  $\mathbf{v}$  is a single variable, then the identity  $\mathbf{u} \approx \mathbf{v}$  is trivial by Lemma 5.1(ii) and so is vacuously deducible from the identities (6.15). Therefore assume that  $\mathbf{u}$  and  $\mathbf{v}$  are words of length at least two. By Lemma 6.18, the identities (6.15) can be used to convert  $\mathbf{u}$  and  $\mathbf{v}$  into words  $\mathbf{u}'$  and  $\mathbf{v}'$  in canonical form. By Lemma 6.19, the words  $\mathbf{u}'$  and  $\mathbf{v}'$  share the same prefix of length two. Therefore  $\mathbf{u}'$  and  $\mathbf{v}'$  are both of the form (CF1) or both of the form (CF2). In any case, it is routinely verified by Lemma 5.1(ii) that  $\mathbf{u}' = \mathbf{v}'$ . Since

$$\mathbf{u} \stackrel{(6.15)}{\approx} \mathbf{u}' = \mathbf{v}' \stackrel{(6.15)}{\approx} \mathbf{v},$$

the identity  $\mathbf{u} \approx \mathbf{v}$  is deducible from (6.15).  $\square$

**Corollary 6.20.** *Suppose that  $S$  is any finite semigroup that satisfies the identities*

$$x^6 \approx x^5, \quad x^5yx \approx x^4yx, \quad xyzt \approx yxzt \quad (6.16)$$

*but violates all of the identities*

$$xy \approx yx, \quad xyz \approx yz. \quad (6.17)$$

*Then  $\langle\langle S \rangle\rangle$  is a subpseudovariety of  $\langle\langle N_5^I, N_2^{\text{bar}} \rangle\rangle$  that is not ji.*

*Proof.* By Proposition 6.17 with  $n = 5$ , the identities satisfied by  $N_5^I \times N_2^{\text{bar}}$  are axiomatized by (6.16). Hence the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle N_5^I, N_2^{\text{bar}} \rangle\rangle = \langle\langle N_5^I \rangle\rangle \vee \langle\langle N_2^{\text{bar}} \rangle\rangle$$

holds. But the two identities in (6.17) are satisfied by  $N_5^I$  and  $N_2^{\text{bar}}$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_5^I \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle N_2^{\text{bar}} \rangle\rangle$ .  $\square$

## 6.6. The pseudovariety $\langle\langle N_2^I, R_2^{\text{bar}} \rangle\rangle$ .

**Proposition 6.21.** *The identities satisfied by the semigroup  $N_2^I \times R_2^{\text{bar}}$  are axiomatized by*

$$x^3 \approx x^2, \quad x^2yx^2 \approx xyx, \quad xhytxy \approx x^2hyty, \quad xhytyx \approx xhy^2tx. \quad (6.18)$$

*Proof.* Let  $S = \{a, b, c, d, e, f\}$  denote the semigroup given by the following multiplication table:

$S$	$a$	$b$	$c$	$d$	$e$	$f$
$a$	$a$	$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$	$a$	$b$	$b$
$c$	$c$	$c$	$c$	$c$	$c$	$c$
$d$	$d$	$d$	$d$	$d$	$d$	$d$
$e$	$a$	$b$	$a$	$d$	$e$	$f$
$f$	$a$	$b$	$d$	$d$	$e$	$f$

The identities satisfied by  $S$  are axiomatized by (6.18) [14, Proposition 26.1]. It is easily deduced from the proof of this result that any identity violated by  $S$  is also violated by one of the following subsemigroups of  $S$ :

$$\{a, d, e\} \cong L_2^I, \quad \{a, b, e\} \cong N_2^I, \quad \{e, f\} \cong R_2, \\ \text{and} \quad \langle c, e, f \rangle = \{a, c, d, e, f\} \cong R_2^{\text{bar}}.$$

Since  $L_2^I, R_2 \in \langle R_2^{\text{bar}} \rangle$ , any identity violated by  $S$  is violated by  $N_2^I$  or  $R_2^{\text{bar}}$ . Therefore  $N_2^I \times R_2^{\text{bar}}$  does not generate any proper subpseudovariety of  $\langle S \rangle$ , whence  $\langle N_2^I \times R_2^{\text{bar}} \rangle = \langle S \rangle$ .  $\square$

**Corollary 6.22.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.18) but violates all of the identities*

$$x^2 \approx x, \quad xy \approx yx. \quad (6.19)$$

*Then  $\langle S \rangle$  is a subpseudovariety of  $\langle N_2^I, R_2^{\text{bar}} \rangle$  that is not ji.*

*Proof.* By Proposition 6.21, the inclusion

$$\langle S \rangle \subseteq \langle N_2^I, R_2^{\text{bar}} \rangle = \langle N_2^I \rangle \vee \langle R_2^{\text{bar}} \rangle$$

holds. But the two identities in (6.19) are satisfied by  $R_2^{\text{bar}}$  and  $N_2^I$ , respectively. Therefore  $\langle S \rangle \not\subseteq \langle N_2^I \rangle$  and  $\langle S \rangle \not\subseteq \langle R_2^{\text{bar}} \rangle$ .  $\square$

#### 6.7. The pseudovariety $\langle L_2^I, \ell_3, \ell_3^{\text{op}} \rangle$ .

**Proposition 6.23** (Zhang and Luo [25, Proposition 3.2(3) and Figure 5]). *The identities satisfied by the semigroup  $L_2^I \times \ell_3 \times \ell_3^{\text{op}}$  are axiomatized by*

$$x^3 \approx x^2, \quad xyx \approx x^2y^2, \quad xy^2z \approx xyz. \quad (6.20)$$

**Corollary 6.24.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.20) but violates all of the identities*

$$x^2y \approx xy, \quad xy^2 \approx xy. \quad (6.21)$$

*Then  $\langle S \rangle$  is a subpseudovariety of  $\langle L_2^I, \ell_3, \ell_3^{\text{op}} \rangle$  that is not ji.*

*Proof.* By Proposition 6.23, the inclusion

$$\langle S \rangle \subseteq \langle L_2^I, \ell_3, \ell_3^{\text{op}} \rangle = \langle L_2^I, \ell_3 \rangle \vee \langle \ell_3^{\text{op}} \rangle$$

holds. But the two identities in (6.21) are satisfied by  $L_2^I \times \ell_3$  and  $\ell_3^{\text{op}}$ , respectively. Therefore  $\langle S \rangle \not\subseteq \langle L_2^I, \ell_3 \rangle$  and  $\langle S \rangle \not\subseteq \langle \ell_3^{\text{op}} \rangle$ .  $\square$

#### 6.8. The pseudovariety $\langle A_0, B_0^I \rangle$ .

**Proposition 6.25** (Lee [7, Proposition 2.8]). *The identities satisfied by the semigroup  $A_0 \times B_0^I$  are axiomatized by*

$$x^3 \approx x^2, \quad x^2yx^2 \approx xyx, \quad xyxy \approx yxyx, \\ xyxzx \approx xyzx, \quad xy^2z^2x \approx xz^2y^2x. \quad (6.22)$$

**Corollary 6.26.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.22) but violates all of the identities*

$$x^2y^2 \approx y^2x^2, \quad xyx \approx yxy. \quad (6.23)$$

*Then  $\langle S \rangle$  is a subpseudovariety of  $\langle A_0, B_0^I \rangle$  that is not ji.*

*Proof.* By Proposition 6.25, the inclusion

$$\langle\langle S \rangle\rangle \subseteq \langle\langle A_0, B_0^I \rangle\rangle = \langle\langle A_0 \rangle\rangle \vee \langle\langle B_0^I \rangle\rangle$$

holds. But the two identities in (6.23) are satisfied by  $B_0^I$  and  $A_0$ , respectively. Therefore  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle A_0 \rangle\rangle$  and  $\langle\langle S \rangle\rangle \not\subseteq \langle\langle B_0^I \rangle\rangle$ .  $\square$

**6.9. The pseudovariety  $\langle\langle W \rangle\rangle$ .** This subsection is concerned with the semigroup  $W = \{a, b, c, d, e\}$  given by the following multiplication table:

$W$	$a$	$b$	$c$	$d$	$e$
$a$	$a$	$a$	$a$	$a$	$a$
$b$	$a$	$a$	$a$	$a$	$c$
$c$	$c$	$c$	$c$	$c$	$c$
$d$	$a$	$b$	$c$	$d$	$a$
$e$	$e$	$e$	$e$	$e$	$e$

**Proposition 6.27.**

(i) *The identities satisfied by the semigroup  $W^I$  are axiomatized by*

$$\begin{aligned} x^3 \approx x^2, \quad xyx^2 \approx xyx, \quad x^2y^2x \approx x^2y^2, \quad xyxy \approx xy^2x, \\ xyhxy \approx xyhyx, \quad xhyxy \approx xhy^2x, \quad xhytxy \approx xhytyx. \end{aligned}$$

(ii)  $\langle\langle W^I \rangle\rangle = \langle\langle W, ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle$ .

*Proof.* (i) This is by Lee and Li [10, Proposition 15.1].

(ii) Since  $(N_2^{\text{bar}})^{\text{op}}$  and the subsemigroup  $\langle b, e \rangle = \{a, b, c, e\}$  of  $W$  are isomorphic, the inclusion  $\langle\langle W, ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle \subseteq \langle\langle W^I \rangle\rangle$  holds. By the proof of part (i) given by Lee and Li [10, Chapter 15], it can be deduced that any identity violated by  $W^I$  is also violated by one of the following subsemigroups of  $W^I$ :

$$\begin{aligned} \{a, c, I\} \cong L_2^I, \quad \{a, b, I\} \cong N_2^I, \quad \langle b, e, I \rangle = \{a, b, c, e, I\} \cong ((N_2^{\text{bar}})^I)^{\text{op}}, \\ \text{and } \langle b, d, e \rangle = \{a, b, c, d, e\} \cong W. \end{aligned}$$

Hence the inclusion  $\langle\langle W^I \rangle\rangle \subseteq \langle\langle L_2^I, N_2^I, ((N_2^{\text{bar}})^I)^{\text{op}}, W \rangle\rangle = \langle\langle W, ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle$  follows.  $\square$

**Lemma 6.28.** *The pseudovariety  $\langle\langle W^I \rangle\rangle$  is not ji.*

*Proof.* By Proposition 6.27(ii), the inclusion

$$\langle\langle W^I \rangle\rangle \subseteq \langle\langle W, ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle = \langle\langle W \rangle\rangle \vee \langle\langle ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle$$

holds. But  $W^I$  violates both the identity  $xyx \approx xy^2$  of  $W$  and the identity  $x^2yx \approx x^2y$  of  $((N_2^{\text{bar}})^I)^{\text{op}}$ . Hence  $\langle\langle W^I \rangle\rangle \not\subseteq \langle\langle W \rangle\rangle$  and  $\langle\langle W^I \rangle\rangle \not\subseteq \langle\langle ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle$ .  $\square$

**Proposition 6.29.**

(i) *The identities satisfied by the semigroup  $W$  are axiomatized by*

$$x^3 \approx x^2, \quad xyx \approx xy^2. \tag{6.24}$$

(ii) *The subpseudovariety of  $\langle\langle W \rangle\rangle$  defined by the identity*

$$x^2 y^2 z^2 \approx x^2 y z^2 \quad (6.25)$$

*is the unique maximal proper subpseudovariety of  $\langle\langle W \rangle\rangle$ .*

**Remark 6.30.** It is routinely shown that the semigroup  $W$  satisfies the identities (6.24) but violates the identity (6.25).

In the present subsection, a word of the form

$$x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m},$$

where  $x_1, x_2, \dots, x_m$  are distinct variables and  $e_1, e_2, \dots, e_m \in \{1, 2\}$ , is said to be in *canonical form*.

**Remark 6.31.** It is easily shown that the identities (6.24) can be used to convert any word into a word in canonical form.

*Proof of Proposition 6.29(i).* As observed in Remark 6.30, the semigroup  $W$  satisfies the identities (6.24). Conversely, suppose that  $\mathbf{u} \approx \mathbf{v}$  is any identity satisfied by  $W$ . By Remark 6.31, the identities (6.24) can be used to convert  $\mathbf{u}$  and  $\mathbf{v}$  into some words  $\mathbf{u}'$  and  $\mathbf{v}'$  in canonical form. Since the subsemigroup  $\{a, c, d\}$  of  $W$  and the semigroup  $L_2^I$  are isomorphic,  $\text{ini}(\mathbf{u}') = \text{ini}(\mathbf{v}')$  by Lemma 5.1(iii). Hence

$$\mathbf{u}' = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{v}' = x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m}$$

for some distinct  $x_1, x_2, \dots, x_m \in \mathcal{A}$  and  $e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m \in \{1, 2\}$ . If  $e_k \neq f_k$ , then by making the substitution  $\varphi$  given by  $x_k \mapsto b$ ,  $x_i \mapsto d$  for all  $i < k$ , and  $x_i \mapsto e$  for all  $i > k$ , the contradiction  $\mathbf{u}'\varphi \neq \mathbf{v}'\varphi$  is obtained. Therefore  $e_i = f_i$  for all  $i$ , so that  $\mathbf{u}' = \mathbf{v}'$ . Since

$$\mathbf{u} \stackrel{(6.24)}{\approx} \mathbf{u}' = \mathbf{v}' \stackrel{(6.24)}{\approx} \mathbf{v},$$

the identity  $\mathbf{u} \approx \mathbf{v}$  is deducible from the identities (6.24).  $\square$

*Proof of Proposition 6.29(ii).* As observed in Remark 6.30, the semigroup  $W$  violates the identity (6.25). Therefore  $\langle\langle W \rangle\rangle \cap \llbracket (6.25) \rrbracket$  is a proper subpseudovariety of  $\langle\langle W \rangle\rangle$ . It remains to verify that every proper subpseudovariety  $\mathbf{V}$  of  $\langle\langle W \rangle\rangle$  satisfies the identity (6.25). Since  $\mathbf{V} \neq \langle\langle W \rangle\rangle$ , there exists an identity  $\mathbf{u} \approx \mathbf{v}$  of  $\mathbf{V}$  that is violated by  $W$ . Further, since the identities (6.24) are satisfied by  $\mathbf{V}$ , it follows from Remark 6.31 that the words  $\mathbf{u}$  and  $\mathbf{v}$  can be chosen to be in canonical form. There are two cases.

CASE 1.  $\text{ini}(\mathbf{u}) \neq \text{ini}(\mathbf{v})$ . Then by Lemma 5.1(iii) and Theorem 5.17, the pseudovariety  $\mathbf{V}$  satisfies the pseudoidentity

$$h^\omega (xh^\omega)^\omega (yh^\omega (xh^\omega)^\omega)^\omega \approx h^\omega (yh^\omega (xh^\omega)^\omega)^\omega.$$

Since

$$h^\omega (xh^\omega)^\omega (yh^\omega (xh^\omega)^\omega)^\omega \stackrel{(6.24)}{\approx} h^2 x^2 y^2 \quad \text{and} \quad h^\omega (yh^\omega (xh^\omega)^\omega)^\omega \stackrel{(6.24)}{\approx} h^2 y^2 x^2,$$

the pseudovariety  $\mathbf{V}$  satisfies the identity

$$h^2 x^2 y^2 \approx h^2 y^2 x^2. \quad (6.26)$$

Since

$$x^2 y z^2 \stackrel{(6.24)}{\approx} x^2 (y z^2)^2 z^2 \stackrel{(6.26)}{\approx} x^2 z^2 (y z^2)^2 \stackrel{(6.24)}{\approx} x^2 z^2 y^2 \stackrel{(6.26)}{\approx} x^2 y^2 z^2,$$

the pseudovariety  $\mathbf{V}$  satisfies the identity (6.25).

CASE 2.  $\text{ini}(\mathbf{u}) = \text{ini}(\mathbf{v})$  and  $\mathbf{u} \neq \mathbf{v}$ . Then

$$\mathbf{u} = x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m} \quad \text{and} \quad \mathbf{v} = x_1^{f_1} x_2^{f_2} \cdots x_m^{f_m}$$

for some distinct  $x_1, x_2, \dots, x_m \in \mathcal{A}$  and  $e_1, e_2, \dots, e_m, f_1, f_2, \dots, f_m \in \{1, 2\}$  such that  $e_k \neq f_k$  for some  $k$ , say  $(e_k, f_k) = (2, 1)$ . Let  $\varphi$  denote the substitution given by  $x_k \mapsto y$ ,  $x_i \mapsto x$  for all  $i < k$ , and  $x_i \mapsto z$  for all  $i > k$ . Then

$$x(\mathbf{u}\varphi)z \stackrel{(6.24)}{\approx} x^2 y^2 z^2 \quad \text{and} \quad x(\mathbf{v}\varphi)z \stackrel{(6.24)}{\approx} x^2 y z^2,$$

so that the pseudovariety  $\mathbf{V}$  satisfies the identity (6.25).  $\square$

**Proposition 6.32.** *The pseudovariety  $\langle\langle W \rangle\rangle$  is sjj but not ji.*

*Proof.* By Proposition 6.29(ii), the pseudovariety  $\langle\langle W \rangle\rangle$  has a unique maximal proper subpseudovariety and so is sjj. If the pseudovariety  $\langle\langle W \rangle\rangle$  is ji, then by Lemma 5.2, the pseudovariety  $\langle\langle W^I \rangle\rangle$  is also ji; but this contradicts Lemma 6.28.  $\square$

**Corollary 6.33.** *Suppose that  $S$  is any finite semigroup that satisfies the identities (6.24) but violates the identity (6.25). Then  $\langle\langle S \rangle\rangle$  coincides with the non-ji pseudovariety  $\langle\langle W \rangle\rangle$ .*

*Proof.* It follows from Proposition 6.29 that  $\langle\langle S \rangle\rangle = \langle\langle W \rangle\rangle$ , a non-ji pseudovariety by Proposition 6.32.  $\square$

## 7. PSEUDOVARIETIES GENERATED BY SEMIGROUPS OF ORDER UP TO FIVE

**Theorem 7.1.** *Suppose that  $S$  is any nontrivial semigroup of order at most five such that the pseudovariety  $\langle\langle S \rangle\rangle$  is ji. Then  $\langle\langle S \rangle\rangle$  coincides with one of the following 30 pseudovarieties:*

$$\begin{array}{llllll} \langle\langle \mathbb{Z}_2 \rangle\rangle, & \langle\langle \mathbb{Z}_3 \rangle\rangle, & \langle\langle \mathbb{Z}_4 \rangle\rangle, & \langle\langle \mathbb{Z}_5 \rangle\rangle, & \langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle, & \langle\langle (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle, \\ \langle\langle N_2 \rangle\rangle, & \langle\langle N_3 \rangle\rangle, & \langle\langle N_4 \rangle\rangle, & \langle\langle N_5 \rangle\rangle, & \langle\langle N_1^I \rangle\rangle, & \langle\langle N_2^I \rangle\rangle, \\ \langle\langle N_3^I \rangle\rangle, & \langle\langle N_4^I \rangle\rangle, & \langle\langle N_2^{\text{bar}} \rangle\rangle, & \langle\langle (N_2^{\text{bar}})^{\text{op}} \rangle\rangle, & \langle\langle (N_2^{\text{bar}})^I \rangle\rangle, & \langle\langle ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle, \\ \langle\langle L_2 \rangle\rangle, & \langle\langle R_2 \rangle\rangle, & \langle\langle L_2^I \rangle\rangle, & \langle\langle R_2^I \rangle\rangle, & \langle\langle L_2^{\text{bar}} \rangle\rangle, & \langle\langle R_2^{\text{bar}} \rangle\rangle, \\ \langle\langle A_0 \rangle\rangle, & \langle\langle A_0^I \rangle\rangle, & \langle\langle A_2 \rangle\rangle, & \langle\langle B_2 \rangle\rangle, & \langle\langle \ell_3^{\text{bar}} \rangle\rangle, & \langle\langle (\ell_3^{\text{bar}})^{\text{op}} \rangle\rangle. \end{array}$$

*Proof.* The 30 pseudovarieties are ji by the following results in Section 5:

Pseudovarieties	Join irreducible by
$\langle\langle \mathbb{Z}_2 \rangle\rangle, \langle\langle \mathbb{Z}_3 \rangle\rangle, \langle\langle \mathbb{Z}_4 \rangle\rangle, \langle\langle \mathbb{Z}_5 \rangle\rangle$	Theorem 5.3
$\langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle, \langle\langle (\mathbb{Z}_2^{\text{bar}})^{\text{op}} \rangle\rangle$	Theorem 5.5
$\langle\langle N_2 \rangle\rangle, \langle\langle N_3 \rangle\rangle, \langle\langle N_4 \rangle\rangle, \langle\langle N_5 \rangle\rangle$	Theorem 5.7
$\langle\langle N_1^I \rangle\rangle, \langle\langle N_2^I \rangle\rangle, \langle\langle N_3^I \rangle\rangle, \langle\langle N_4^I \rangle\rangle$	Theorem 5.9
$\langle\langle N_2^{\text{bar}} \rangle\rangle, \langle\langle (N_2^{\text{bar}})^{\text{op}} \rangle\rangle$	Theorem 5.11
$\langle\langle (N_2^{\text{bar}})^I \rangle\rangle, \langle\langle ((N_2^{\text{bar}})^I)^{\text{op}} \rangle\rangle$	Theorem 5.13
$\langle\langle L_2 \rangle\rangle, \langle\langle R_2 \rangle\rangle$	Theorem 5.15
$\langle\langle L_2^I \rangle\rangle, \langle\langle R_2^I \rangle\rangle$	Theorem 5.17
$\langle\langle L_2^{\text{bar}} \rangle\rangle, \langle\langle R_2^{\text{bar}} \rangle\rangle$	Theorem 5.19
$\langle\langle A_0 \rangle\rangle$	Theorem 5.21
$\langle\langle A_0^I \rangle\rangle$	Theorem 5.23
$\langle\langle A_2 \rangle\rangle$	Theorem 5.25
$\langle\langle B_2 \rangle\rangle$	Theorem 5.27
$\langle\langle \ell_3^{\text{bar}} \rangle\rangle, \langle\langle (\ell_3^{\text{bar}})^{\text{op}} \rangle\rangle$	Theorem 5.29

Up to isomorphism and anti-isomorphism, there exist 1308 nontrivial semigroups of order up to five. With the aid of a computer, it is routinely determined, using the sufficient conditions given in Subsections 7.1 and 7.2 below, which of these semigroups generate ji pseudovarieties. Specifically, by Conditions A1–A23 and their dual conditions, 241 of the 1308 semigroups generate the ji pseudovarieties listed in Theorem 7.1; by Conditions B1–B13 and their dual conditions, 1067 of the 1308 semigroups generate pseudovarieties that are not ji. The proof of Theorem 7.1 is thus complete.  $\square$



The following table lists, up to isomorphism and anti-isomorphism, the number of semigroups of each order  $n \in \{2, 3, 4, 5\}$  that generate ji pseudovarieties:

	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$2 \leq n \leq 5$
Number of semigroups of order $n$ generating ji pseudovarieties	4	8	33	196	241
Number of semigroups of order $n$ generating non-ji pseudovarieties	0	10	93	964	1067
Total number of semigroups of order $n$	4	18	126	1160	1308

**7.1. Conditions sufficient for join irreducibility.** The following conditions and their dual conditions are sufficient for a finite semigroup  $S$  to generate a ji pseudovariety in Theorem 7.1.

**Condition A1** (Proposition 5.4). *The equality  $\langle\langle S \rangle\rangle = \langle\langle \mathbb{Z}_2 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^2y \approx y\}$ ,
- $S \not\models x \approx y$ .

**Condition A2** (Proposition 5.4). *The equality  $\langle\langle S \rangle\rangle = \langle\langle \mathbb{Z}_3 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^3y \approx y\}$ ,
- $S \not\models x \approx y$ .

**Condition A3** (Proposition 5.4). *The equality  $\langle\langle S \rangle\rangle = \langle\langle \mathbb{Z}_4 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^4y \approx y\}$ ,
- $S \not\models x^3 \approx x$ .

**Condition A4** (Proposition 5.4). *The equality  $\langle\langle S \rangle\rangle = \langle\langle \mathbb{Z}_5 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^5y \approx y\}$ ,
- $S \not\models x \approx y$ .

**Condition A5** (Proposition 5.6). *The equality  $\langle\langle S \rangle\rangle = \langle\langle \mathbb{Z}_2^{\text{bar}} \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x, xyxy \approx yx^2y\}$ ,
- $S \not\models xyx \approx yx^2$ .

**Condition A6** (Proposition 5.8). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_2 \rangle\rangle$  holds if*

- $S \models x^2 \approx y_1y_2$ ,
- $S \not\models x^2 \approx x$ .

**Condition A7** (Proposition 5.8). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_3 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^3 \approx y_1y_2y_3\}$ ,
- $S \not\models x^3 \approx x^2$ .

**Condition A8** (Proposition 5.8). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_4 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^2y \approx xy^2, x^4 \approx y_1y_2y_3y_4\}$ ,
- $S \not\models x^4 \approx x^3$ .

**Condition A9** (Proposition 5.8). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_5 \rangle\rangle$  holds if*

- $S \models \{xy \approx yx, x^2yz \approx xy^2z, x^5 \approx y_1y_2y_3y_4y_5\},$
- $S \not\models x^5 \approx x^4.$

**Condition A10** (Proposition 5.10). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_1^I \rangle\rangle$  holds if*

- $S \models \{x^2 \approx x, xy \approx yx\},$
- $S \not\models x \approx y.$

**Condition A11** (Proposition 5.10). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_2^I \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, xy \approx yx\},$
- $S \not\models x^2y \approx xy^2.$

**Condition A12** (Proposition 5.10). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_3^I \rangle\rangle$  holds if*

- $S \models \{x^4 \approx x^3, xy \approx yx\},$
- $S \not\models x^3y^2 \approx x^2y^3.$

**Condition A13** (Proposition 5.10). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_4^I \rangle\rangle$  holds if*

- $S \models \{x^5 \approx x^4, xy \approx yx\},$
- $S \not\models x^4y^3 \approx x^3y^4.$

**Condition A14** (Proposition 5.12). *The equality  $\langle\langle S \rangle\rangle = \langle\langle N_2^{\text{bar}} \rangle\rangle$  holds if*

- $S \models xyz \approx yz,$
- $S \not\models xy \approx y^2.$

**Condition A15** (Proposition 5.14). *The equality  $\langle\langle S \rangle\rangle = \langle\langle (N_2^{\text{bar}})^I \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, x^2hx \approx xhx, xhx^2 \approx hx^2, xhxtx \approx hxtx,$   
 $xyxy \approx yx^2y, xyhxy \approx yxhxy, xyxty \approx yx^2ty, xyhxy \approx yxhxy\},$
- $S \not\models xyxyh^2 \approx x^2y^2h^2.$

**Condition A16** (Proposition 5.16). *The equality  $\langle\langle S \rangle\rangle = \langle\langle L_2 \rangle\rangle$  holds if*

- $S \models xy \approx x,$
- $S \not\models x \approx y.$

**Condition A17** (Proposition 5.18). *The equality  $\langle\langle S \rangle\rangle = \langle\langle L_2^I \rangle\rangle$  holds if*

- $S \models \{x^2 \approx x, xyx \approx xy\},$
- $S \not\models xyz \approx xzy.$

**Condition A18** (Proposition 5.20). *The equality  $\langle\langle S \rangle\rangle = \langle\langle L_2^{\text{bar}} \rangle\rangle$  holds if*

- $S \models \{x^2 \approx x, xyz \approx xzxyz\},$
- $S \not\models yxzx \approx xyzx.$

**Condition A19** (Proposition 5.22). *The equality  $\langle\langle S \rangle\rangle = \langle\langle A_0 \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, xyx \approx xyxy, xyx \approx yxyx\},$
- $S \not\models x^2y^2 \approx y^2x^2.$

**Condition A20** (Proposition 5.24). *The equality  $\langle\langle S \rangle\rangle = \langle\langle A_0^I \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, x^2yx^2 \approx xyx, xyxy \approx yxyx, yxzx \approx xyxz\},$
- $S \not\models hx^2y^2h \approx hy^2x^2h.$

**Condition A21** (Proposition 5.26). *The equality  $\langle\langle S \rangle\rangle = \langle\langle A_2 \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, xyxyx \approx yxy, yxzx \approx xzyx\},$
- $S \not\models x^2y^2x^2 \approx x^2yx^2.$

**Condition A22** (Proposition 5.28). *The equality  $\langle\langle S \rangle\rangle = \langle\langle B_2 \rangle\rangle$  holds if*

- $S \models \{x^3 \approx x^2, xyxyx \approx yx, x^2y^2 \approx y^2x^2\},$
- $S \not\models xy^2x \approx yx.$

**Condition A23** (Proposition 5.30). *The equality  $\langle\langle S \rangle\rangle = \langle\ell_3^{\text{bar}}\rangle$  holds if*

- $S \models \{x^2y \approx xy, xyz \approx yxyz\},$
- $S \not\models xyzx \approx yxzx.$

**7.2. Conditions sufficient for non-join irreducibility.** The following conditions and their dual conditions are sufficient for a finite semigroup  $S$  to generate a pseudovariety that is not ji.

**Condition B1** (Corollary 6.2). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}}, N_3^I\rangle$  if*

- $S \models \{x^{15} \approx x^3, x^{14}hx \approx x^2hx, x^{13}hxtx \approx xhxtx, x^3hx \approx xhx^3,$   
 $hx^2tx \approx x^3htx, hx^2y^2ty \approx xhy^2x^2ty,$   
 $xykxytxdy \approx xhykxytxdy, xhykxytydx \approx xhykxytydx\},$
- $S \not\models x^3 \approx x, \quad S \not\models xy \approx yx.$

**Condition B2** (Corollary 6.3). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}}, N_3^I\rangle$  if*

- $S \models \{x^{15} \approx x^3, x^{14}hx \approx x^2hx, x^{13}hxtx \approx xhxtx, x^3hx \approx xhx^3,$   
 $hx^2tx \approx x^3htx, hx^2y^2ty \approx xhy^2x^2ty,$   
 $xykxytxdy \approx xhykxytxdy, xhykxytydx \approx xhykxytydx\},$
- $S \not\models xy \approx yx, \quad S \not\models xyx^2 \approx xy, \quad S \not\models x^2yx \approx yx.$

**Condition B3** (Corollary 6.4). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2^{\text{bar}}, (\mathbb{Z}_2^{\text{bar}})^{\text{op}}, N_3^I\rangle$  if*

- $S \models \{x^{15} \approx x^3, x^{14}hx \approx x^2hx, x^{13}hxtx \approx xhxtx, x^3hx \approx xhx^3,$   
 $hx^2tx \approx x^3htx, hx^2y^2ty \approx xhy^2x^2ty,$   
 $xykxytxdy \approx xhykxytxdy, xhykxytydx \approx xhykxytydx\},$
- $S \not\models x^4 \approx x^3, \quad S \not\models x^4y \approx x^4, \quad S \not\models x^3y \approx x^3, \quad S \not\models xyx^2 \approx xy,$   
 $S \not\models x^2yx \approx yx.$

**Condition B4** (Corollary 6.10). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\mathbb{Z}_6, N_5^I, L_2^I, R_2^I, A_0^I\rangle$  if*

- $S \models \{x^{11} \approx x^5, x^{10}yx \approx x^4yx, x^2yx \approx xyx^2, xyxzx \approx x^2yzx\},$
- $S \not\models x^2 \approx x, \quad S \not\models xyxy \approx yxyx.$

**Condition B5** (Corollary 6.11). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\mathbb{Z}_6, N_5^I, L_2^I, R_2^I, A_0^I\rangle$  if*

- $S \models \{x^{11} \approx x^5, x^{10}yx \approx x^4yx, x^2yx \approx xyx^2, xyxzx \approx x^2yzx\},$
- $S \not\models x^6 \approx x^5, \quad S \not\models x^6y \approx y.$

**Condition B6** (Corollary 6.13). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji pseudovariety of nilpotent semigroups if*

- $S \models x^6 \approx y_1y_2y_3y_4y_5y_6,$
- $S \not\models xy \approx yx.$

**Condition B7** (Corollary 6.15). *A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle N_4, N_2^I \rangle$  if*

- $S \models \{xy \approx yx, x^3y_1y_2 \approx x^2y_1y_2\},$
- $S \not\models x^3 \approx x^2, \quad S \not\models x^2y \approx xy^2.$

**Condition B8** (Corollary 6.16). A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\langle N_5, N_1^I \rangle\rangle$  if

- $S \models \{xy \approx yx, x^2yz \approx xy^2z, x^2y_1y_2y_3y_4 \approx xy_1y_2y_3y_4\},$
- $S \not\models x^2 \approx x, \quad S \not\models x^5 \approx y^5.$

**Condition B9** (Corollary 6.20). A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\langle N_5^I, N_2^{\text{bar}} \rangle\rangle$  if

- $S \models \{x^6 \approx x^5, x^5yx \approx x^4yx, xyzt \approx yxzt\},$
- $S \not\models xy \approx yx, \quad S \not\models xyz \approx yz.$

**Condition B10** (Corollary 6.22). A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\langle N_2^I, R_2^{\text{bar}} \rangle\rangle$  if

- $S \models \{x^3 \approx x^2, x^2yx^2 \approx xyx, xhytxy \approx x^2hyty, xhytyx \approx xhy^2tx\},$
- $S \not\models x^2 \approx x, \quad S \not\models xy \approx yx.$

**Condition B11** (Corollary 6.24). A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\langle L_2^I, \ell_3, \ell_3^{\text{op}} \rangle\rangle$  if

- $S \models \{x^3 \approx x^2, xyx \approx x^2y^2, xy^2z \approx xyz\},$
- $S \not\models x^2y \approx xy, \quad S \not\models xy^2 \approx xy.$

**Condition B12** (Corollary 6.26). A pseudovariety  $\langle\langle S \rangle\rangle$  is a non-ji subpseudovariety of  $\langle\langle A_0, B_0^I \rangle\rangle$  if

- $S \models \{x^3 \approx x^2, x^2yx^2 \approx xyx, xyxy \approx yxyx, xyxzx \approx xyzx, \\ xy^2z^2x \approx xz^2y^2x\},$
- $S \not\models x^2y^2 \approx y^2x^2, \quad S \not\models xyx \approx yxy.$

**Condition B13** (Corollary 6.33). A pseudovariety  $\langle\langle S \rangle\rangle$  coincides with the non-ji pseudovariety  $\langle\langle W \rangle\rangle$  if

- $S \models \{x^3 \approx x^2, xyx \approx xy^2\},$
- $S \not\models x^2y^2z^2 \approx x^2yz^2.$

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